Chair of Structural Mechanics
TUM School of Engineering and Design
Technical University of Munich

Master's Thesis

Exploring Bloch Waves in Periodic Media using the Plane Wave Expansion Method

Wave Propagation in Periodic Media – Background and Motivation

In structures with a periodically repeating geometry or material distribution, wave propagation is no longer governed by a single speed of sound: The periodicity fundamentally alters how waves propagate through the structure. The pivotal element to reveal the wave propagation features is the *dispersion relation*, which links the frequency and wavenumber and describes how the eigenfrequencies of the system evolve as a function of their spatial wavelength. [1]

From the dispersion relation, one can directly infer whether waves at a given frequency propagate freely, are slowed down, or are completely attenuated in certain frequency bands (so-called *band gaps*). These wave filtering properties render periodic structures particularly attractive for vibroacoustic applications as they can provide adjustable mitigation of vibrations without relying solely on heavy damping or added mass. By carefully tailoring the geometry and material layout of the repeating unit, one may shape the dispersion relation and thereby tune the wave filtering properties of the periodic structure. Numerical prediction of dispersion relations is, therefore, a central step in the design of structures for wave control. [2]

There exists a variety of computational techniques for numerically studying the dispersion relation of a periodic structure. As an example, consider the longitudinal motion of a one-dimensional periodic rod with a spatially varying Young's modulus $E(x)$ and material density $\rho(x)$. The governing equation for time-harmonic waves is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[E(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right] + \rho(x)\,\omega^2 u = 0, \tag{1}$$

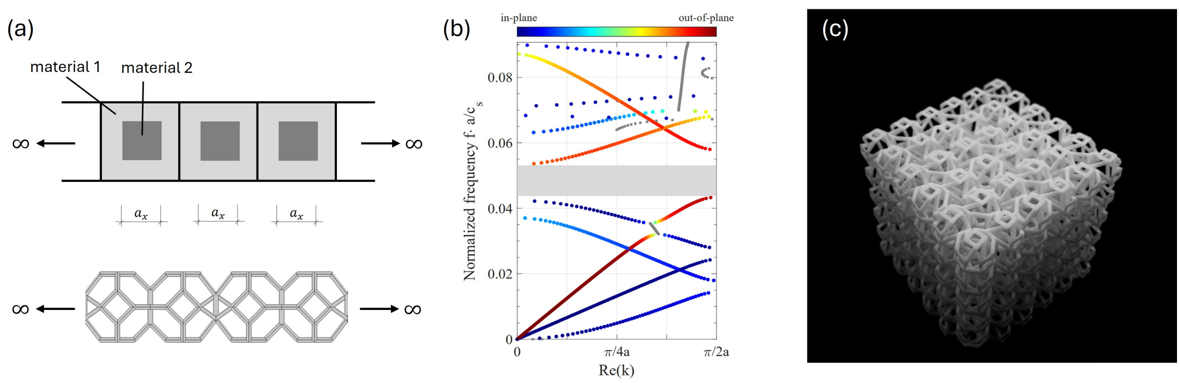

Figure 1: Illustrative examples of periodic structures: (a) one-dimensional periodic media with material (top) and geometric periodicity (bottom), (b) a representative dispersion relation showing a band gap (gray area), and (c) an additively manufactured structure designed for wave-control applications.

where $E(x)$ and $\rho(x)$ are periodic functions with period a. A fundamental consequence of periodicity is that the solutions to this equation take the form of BLOCH waves

$$u_k(x) = \tilde{u}_k(x)\mathrm{e}^{\mathrm{i}kx}, \qquad \tilde{u}_k(x+\mathrm{a}) = \tilde{u}_k(x). \tag{2}$$

Here, k denotes the wavenumber and $\tilde{u}_k(x)$ are the BLOCH functions that modulate the plane wave solutions with the periodicity a of the underlying structure [3]. A conventional way to solve such problems is to employ finite element methods: The domain is spatially discretized, the wave field $u(x)$ is approximated using suitable shape functions, and the BLOCH-type character of the waves is enforced through periodic boundary conditions. This approach leads to a large but sparse matrix eigenvalue problem

$$\left[\mathbf{M} - \omega^2 \mathbf{K}(k)\right] \tilde{\mathbf{u}}_{\mathbf{k}} = 0, \tag{3}$$

whose eigensolutions $\omega(k)$ encode the dispersion relation, also called *band structure.*

In contrast, the *Plane Wave Exanpsion* method approaches the same problem in the spectral domain and exploits the periodicity by expanding both the sought-after solution field and the material coefficients into FOURIER series. Given the rod example, the material coefficients are expanded by

$$E(x) = \sum_{m=-\infty}^{\infty} E_m \, e^{\mathrm{i}m\mathrm{G}x}, \qquad \rho(x) = \sum_{m=-\infty}^{\infty} \rho_m \, e^{\mathrm{i}m\mathrm{G}x} \quad \text{with} \quad \mathrm{G} = \frac{2\pi}{\mathrm{a}}, \tag{4}$$

while the periodic part of the BLOCH solution is likewise expanded by

$$u_k(x) = \sum_{l=-\infty}^{\infty} \hat{u}_l \, e^{\mathrm{i}(k+l\mathrm{G})x}. \tag{5}$$

Substituting these expansions into the wave equation (1) yields the Plane Wave Expansion system

$$\sum_{l=-\infty}^{\infty} \left[(k+n\mathrm{G})(k+l\mathrm{G})\, E_{n-l} - \omega^2 \rho_{n-l}\right] \hat{u}_m = 0, \tag{6}$$

which is the FOURIER-domain representation of the dispersion eigenvalue problem. By truncating the expansions to a finite number of FOURIER harmonics, one obtains the dispersion relation $\omega(k)$ by solving the eigenvalue problem for a sequence of wavenumbers k.

Compared to conventional finite element (FE) simulations, the PWE method does not rely on meshing the physical domain and inherently satisfies BLOCH's theorem without the need for periodic boundary conditions. Moreover, they offer a more intuitive approach of how the dispersion relation is shaped by coupling of the FOURIER components, which offers a physically meaningful interpretation of various dispersion phenomena. At the same time, PWE methods require careful implementation and convergence studies, and are restricted to ideal, infinite periodic media. [4]

Key Research Questions

- How can the PWE method be derived, implemented and validated for 1D periodic structures?
- What are the advantages, limitations, and convergence properties of PWE compared to FE?
- How do symmetries (translation, reflection, glide/screw) manifest in the Fourier-domain?
- Can locally resonant inclusions be included in the PWE framework?

Project Tasks and Stages

1. **Analytical foundation: Mass-Spring Chains**
   Derive dispersion relations for monoatomic and diatomic chains, visualize Brillouin zones, and interpret band structures.
2. **Literature study on PWE and Bloch waves**
   Study Bloch's theorem, Fourier expansions, and numerical methods for periodic media.
3. **Implementation of a PWE solver**
   Implement PWE for a 1D periodic rod (MATLAB/Python), verify against analytical models, and perform convergence studies.
4. **Validation and Comparison**
   Compute band structures using PWE and FE (COMSOL Multiphysics) and compare results for accuracy and efficiency.
5. **Optional Extensions**
   Investigate effects of symmetries, add local resonances, and extend the PWE formulation for complex-valued wavenumbers.

Recommended Background and Skills

The topic of this thesis is most suitable for students who

- have a solid background in **structural dynamics**,
- are familiar with **finite element** and **integral transform methods**,
- have strong programming skills in **MATLAB** or **Python**,
- and enjoy working **very independently** and **systematically**, including carefully studying the relevant scientific literature

Students with a keen interest in vibroacoustics, solid state physics, and computational methods will find this topic particularly interesting.

Supervisor

Lukas Kleine-Wächter (Room N1150), lukas.kleinewaechter@tum.de

References

[1] Deymier, Pierre A. *Acoustic Metamaterials and Phononic Crystals*, volume 173. Springer Berlin Heidelberg, 2013. `doi:10.1007/978-3-642-31232-8`.

[2] Mahmoud I. Hussein, Michael J. Leamy, and Massimo Ruzzene. Dynamics of phononic materials and structures: Historical origins, recent progress, and future outlook. *Applied Mechanics Reviews*, 66(4):73, 2014. `doi:10.1115/1.4026911`.

[3] J. D. Joannopoulos. *Photonic crystals: Molding the flow of light*. Princeton University Press, Princeton, 2nd edition, 2008.

[4] Vincent Laude. *Phononic Crystals*. De Gruyter, Berlin, Boston, 2020. `doi:10.1515/9783110641189`.