Industrial Applications of Computational Mechanics Extended Civil Engineering Problems

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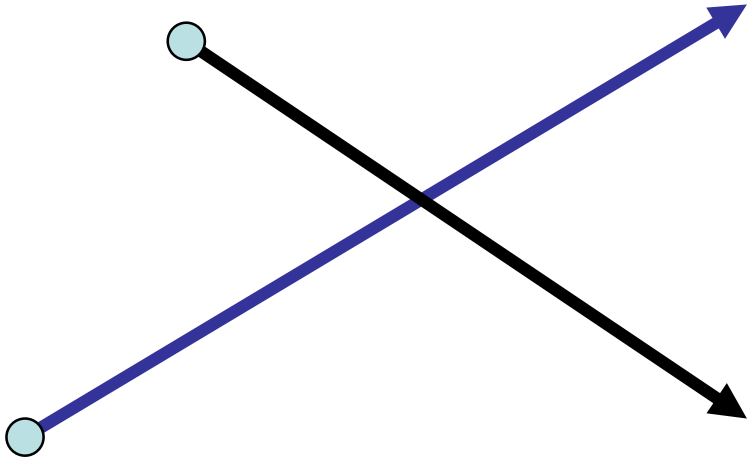
Beyond Structural Analysis

- **Multiphysics**
  - » **Heat Conduction, Radiation**
  - » **Potential Flow (Laplace)**
  - » **Fluid mechanics (Navier Stokes)**
  - » **Wind engineering**
  - » **Fire and Blast**
  - » **Combustion**

Mathematical Excursion



$$g1 := o1 + \alpha \cdot d1$$

$$g2 := o2 + \beta \cdot d2$$

Intersection of two lines in 3D

- g1 = g2

$$x_1 + \alpha \cdot dx_1 \; - x_2 + \beta \cdot dx_2 = 0$$

$$y_1 + \alpha \cdot dy_1 \; - y_2 + \beta \cdot dy_2 = 0$$

$$z_1 + \alpha \cdot dz_1 \; - z_2 + \beta \cdot dz_2 = 0$$

- 3 equations for 2 unknowns α, β
- You need some case constructs to select the optimal solution strategy !

Alternate approach

- **Distance (g1 - g2) = Minimum**

$$(x_1 + \alpha \cdot dx_1 - x_2 - \beta \cdot dx_2)^2 +$$

$$(y_1 + \alpha \cdot dy_1 - y_2 - \beta \cdot dy_2)^2 +$$

$$(z_1 + \alpha \cdot dz_1 - z_2 - \beta \cdot dz_2)^2 = A^2$$

- **2 equations for 2 unknowns α, β**

$$\partial A / \partial \alpha = 0$$

$$\partial A / \partial \beta = 0$$

Minimum Approach

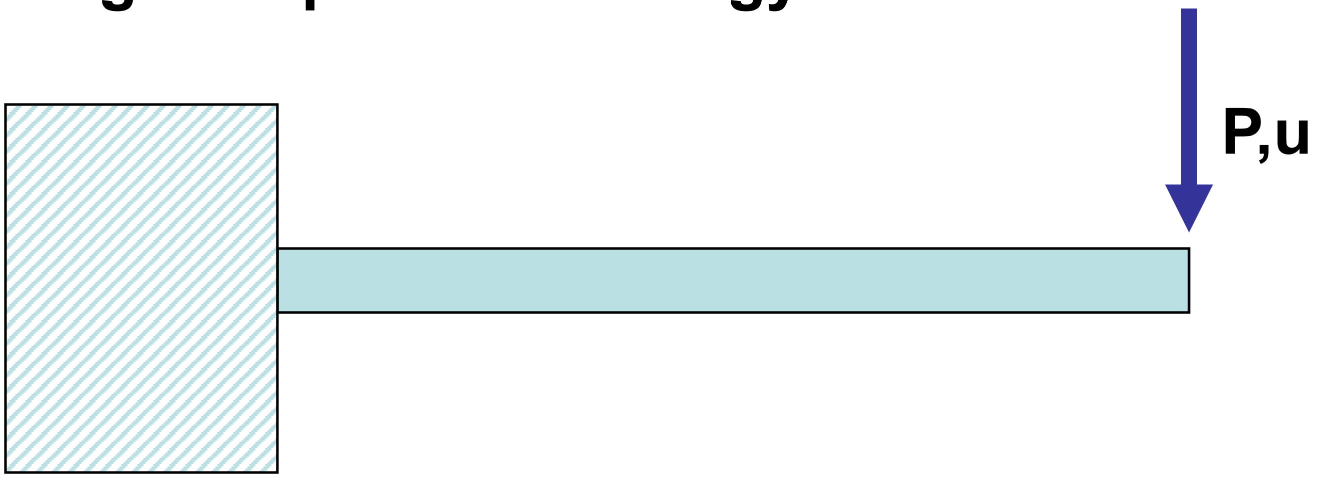
$$\begin{bmatrix} dx_1^2 + dy_1^2 + dz_1^2 & dx_1dx_2 + dy_1dy_2 + dz_1dz_2 \\ sym & dx_2^2 + dy_2^2 + dz_2^2 \end{bmatrix} \bullet \begin{bmatrix} \alpha \\ \beta \end{bmatrix} =$$

$$\begin{bmatrix} (x_1 - x_2)dx_1 + (y_1 - y_2)dy_1 + (z_1 - z_2)dz_1 \\ (x_1 - x_2)dx_2 + (y_1 - y_2)dy_2 + (z_1 - z_2)dz_2 \end{bmatrix}$$

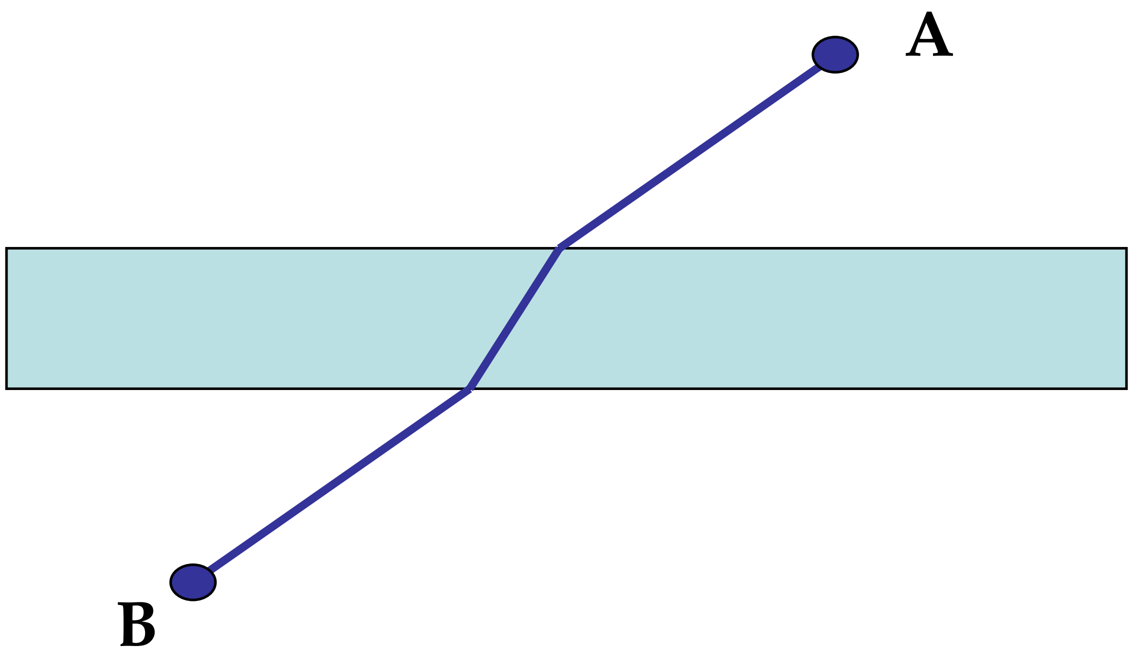
- **16 lines of code instead of 200**
- **Only two conditions to check**
  - » **Determinant becomes zero if and only if lines are parallel**
  - » **Resulting distance gives skewed lines problem for free**

Minimum of Energy

- Each mechanical system will deform in such a way that the total energy will become a minimum.
- Positive deformation Energy $1/2 \cdot u^T \cdot K \cdot u$
- Negative potential Energy $P \cdot u$



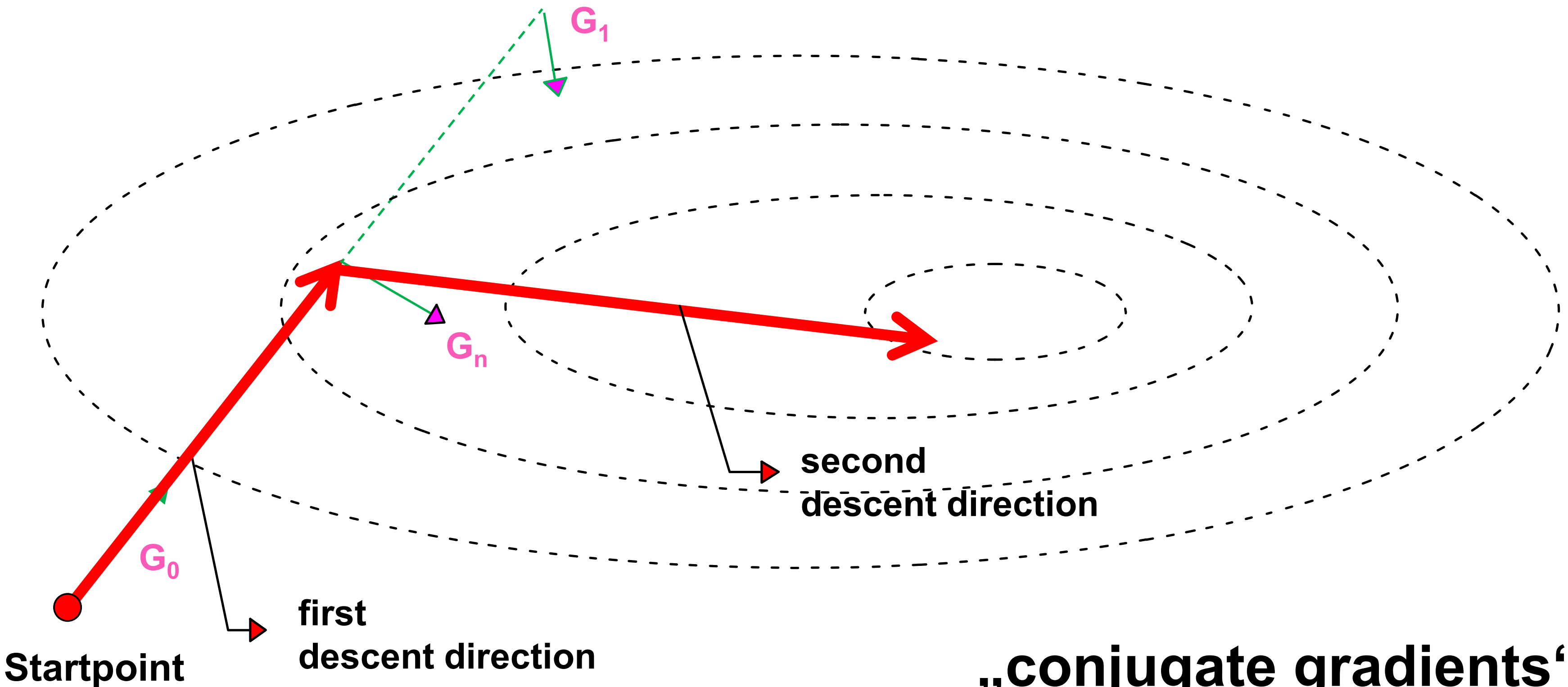
Refraction of light (QED)



What is the fastest (most probable) way for the light from A to B if the speed in the blue body is slower ?

Optimization and Line-Search

$$\min \Pi \quad \Leftrightarrow \quad \frac{\partial \quad \Pi}{\partial \quad x} = 0$$



„conjugate gradients“

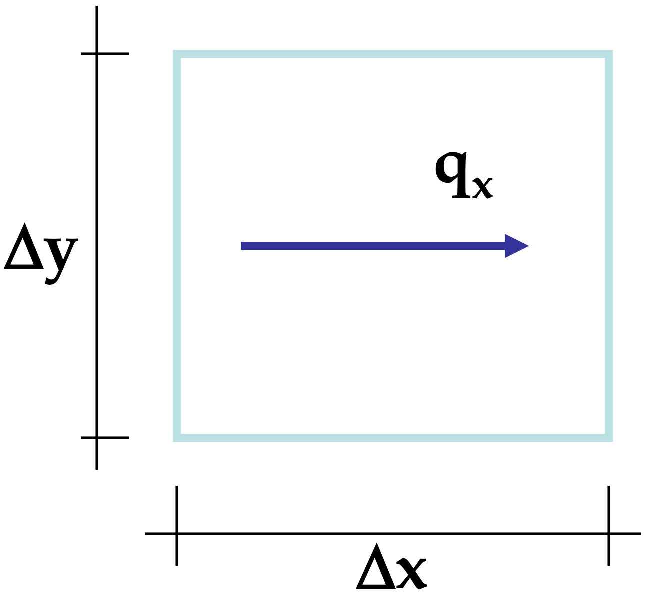
Heat Conduction 2D

Requested:
Temperature field T(x,y)

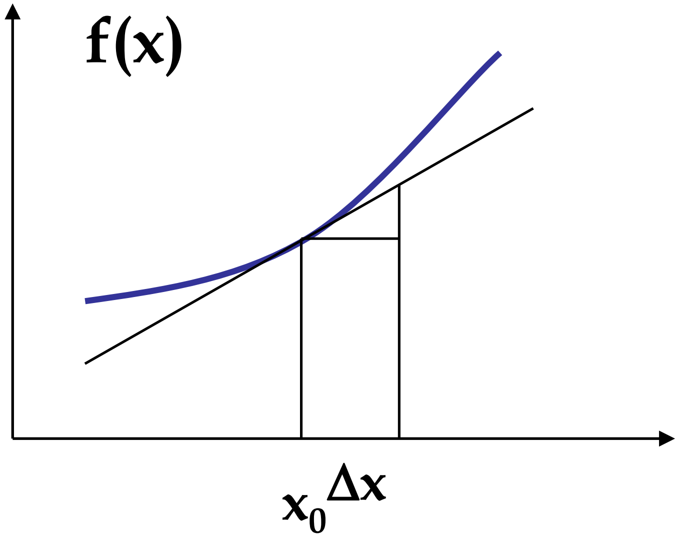
Heat conductivity k

$$q_x = k \cdot \frac{\Delta T}{\Delta x} \Rightarrow q_x = k \cdot \frac{\partial T}{\partial x}$$

$$q_y = k \cdot \frac{\Delta T}{\Delta y} \Rightarrow q_y = k \cdot \frac{\partial T}{\partial y}$$



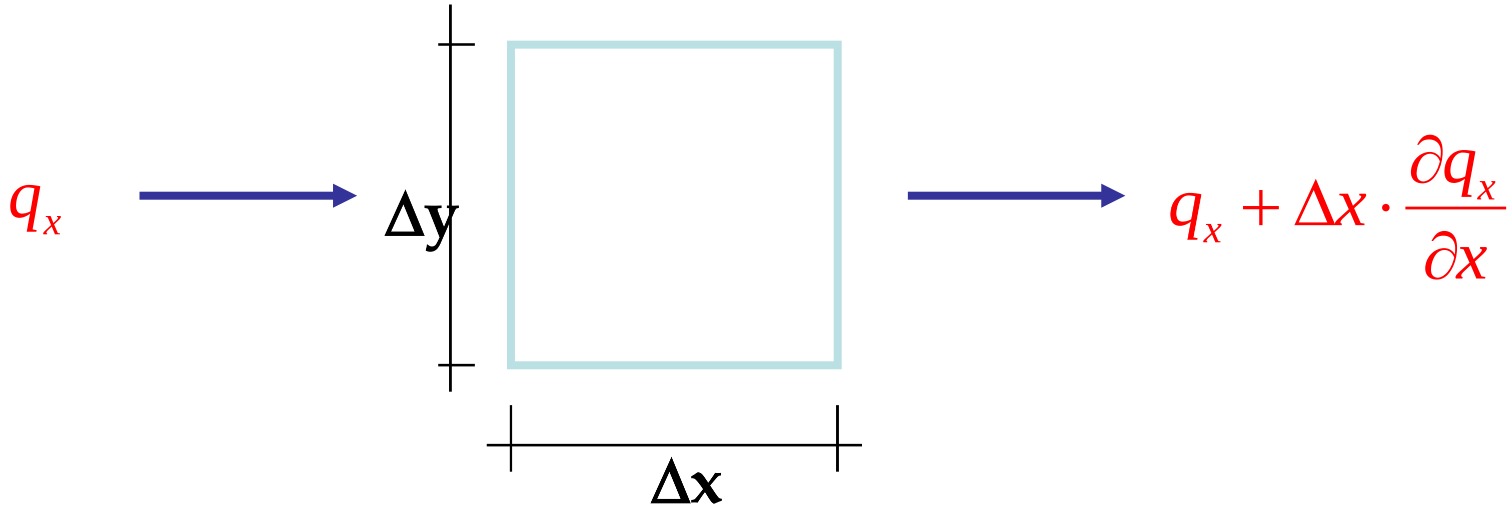
Taylor Series



$$f(x+\Delta x) = f(x_0) + \Delta x \cdot \frac{df(x_0)}{dx} + \frac{\Delta x^2}{2!} \cdot \frac{d^2 f(x_0)}{dx^2} + \ldots$$

$$(\,for\ \Delta x \rightarrow 0 \quad all\ contributions\ of\ higher\ order\ will\ vanish!)$$

Heat quantity conservation



$$\Delta y \cdot \left(q_x + \Delta x \cdot \frac{\partial q_x}{\partial x} - q_x \right) + \Delta x \cdot \left(q_y + \Delta y \cdot \frac{\partial q_y}{\partial y} - q_y \right) = 0$$

Laplace Equation

$$\Delta y \cdot \left(q_x + \Delta x \cdot \frac{\partial q_x}{\partial x} - q_x \right) + \Delta x \cdot \left(q_y + \Delta y \cdot \frac{\partial q_y}{\partial y} - q_y \right) = 0$$

$$q_x = k \cdot \frac{\partial T}{\partial x} \quad q_y = k \cdot \frac{\partial T}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(k \cdot \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k \cdot \frac{\partial T}{\partial y} \right) = 0$$

$$\Rightarrow k \cdot \Delta T = 0$$

The complete picture

$$div\left(k \cdot grad\, T\right) + S\frac{\partial T}{\partial t} - q = 0$$

- Where
  - » T is potential value (e.g. Temperature)
  - » k is conductivity
  - » S is the storage coefficient
  - » q is a source term within the domain
- And the basic boundary conditions
  - » $T = T_1$ on Γ_1
  - » $u^t \cdot n = q_2 + \alpha\,(T - T_2)$ on Γ_2

Fields and Gradients

- **A scalar value U which is defined in every point of space with a distinct value (eg. Temperature) is called a scalar field.**
- **A value u which is defined in every point of space as a vector (e.g. flow velocities) is called a vector field.**
- **A partial derivative of a scalar field U defines a vector field:**

$$u = grad\,U = \begin{bmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y} \\ \frac{\partial U}{\partial z} \end{bmatrix}$$

Divergence

- **Differentiating a vector field we have the divergence and the curl or rotation given by:**

$$div\, u = \nabla U = \left\lfloor \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right\rfloor$$

$$rot\, u = \left\lfloor \begin{matrix} \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \\ \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \end{matrix} \right\rfloor$$

Potential Fields

- **A conservative field or potential field is a vector field, where the integral along a curve $\int u dr$ depends only on the start point A and the end point B, but not on the integration path.**
- **A conservative field is free of vortices, i.e. the closed integral with A=B is zero.**
- **If sources or sinks are present in the system however the closed integral will be equal to the sum of sources (vortices).**
- **A conservative field has its largest and smallest value at the boundary (Maximum principle).**

Uniqueness of Solutions

- There are many cases where a quantity is only as a derivative within the equations. Then its value is only defined by the selection of a reference value in any arbitrary point.
  - » e.g. displacements for rigid body movements
  - » e.g. pressure for a flow problem
- As the fundamental solution (Greens function) of a 2D problem is the logarithm function, while in 3D it is the 1/r function, there is an important difference.
  In 3D we may easily select a zero potential at infinity, for 2D we have to select this point within a finite distance. This makes 2D solutions in many cases less unique than 3D solutions.

Analytical Solutions of the Laplace-Equation

- All functions of a complex Variable will solve the Laplace equation with their real and their imaginary part (conformal mapping).

$$F(z) = g(z) + i * h(z)$$

e.g.

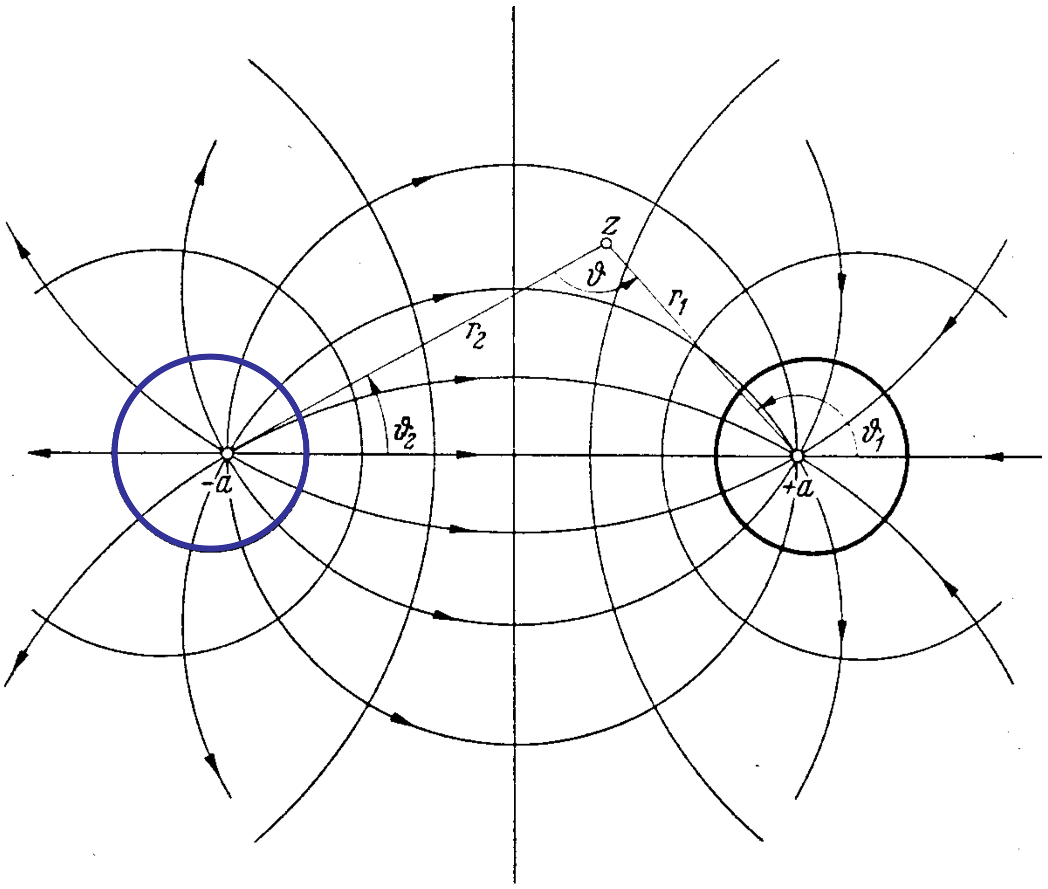
$$Ln(z) = \ln(\sqrt{x^2+y^2}) + i * \operatorname{atan}(y/x)$$

- In 3D the fundamental solution is 1/r but there are no equivalent techniques available

Two Point Flow

- **Combination of functions will yield more solutions:**

$$\frac{1}{2\pi}\ln\left(\frac{z+a}{z-a}\right)$$



Laplace Solutions

- **Principal Problem: <u>Boundary Conditions</u>**
  Beside some special cases it is impossible to find a solution fullfilling the boundary conditions.
- **General Solution Possibility:**
  Use a limited but large set of fundamental solutions and try to full fill the boundary conditions point wise or with a weighted approach approximately.
  Example: Integral equations

Alternate Formulations

- **Variational Approach**
  Integral over the domain Ω and the boundary Γ

$$\int\int k \cdot \left(\frac{\partial T}{\partial x}\right)^2 + k \cdot \left(\frac{\partial T}{\partial y}\right)^2 d\Omega - \int q \cdot T d\Gamma = Minimum$$

- **Method of weighted Residuals**

$$\int\int W \cdot \Big(L(u) - L(u)\Big) d\Omega - \int w \cdot \Big(R(u) - R(u)\Big) d\Gamma = Min$$

The complete picture

$$\frac{1}{2}\int_{\Omega} k_{ij}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}d\Omega+\int_{\Omega} T\left(S\frac{\partial T}{\partial t}-q\right)d\Omega$$

$$+\int_{\Gamma_1}\left(T-T_1\right)k_{ij}\frac{\partial T}{\partial x_j}d\Gamma_1-\int_{\Gamma_2} T\cdot\left(q-\frac{1}{2}\alpha T\right)d\Gamma_2$$

- **Remarks**
  - » **The function T is selected best to fulfil the Dirichlet boundary condition $T=T_1$ on Γ_1 in advance**
  - » **On all boundaries without an explicit integral we will obtain the natural boundary condition $u^t \cdot n = 0$**
  - » **The transient term is only correct if we assume the time derivative to be constant**

Numerical variational analysis

- **Instead of the unknown function we use a series of known functions with variable parameters and integrate the functional (analytically or even numerically)**

$$y = \sum_{i=1}^{n} a_i \cdot N_i(x)$$

$$\prod = \prod(a_1, a_2, \ldots, a_n)$$

Numerical variational analysis

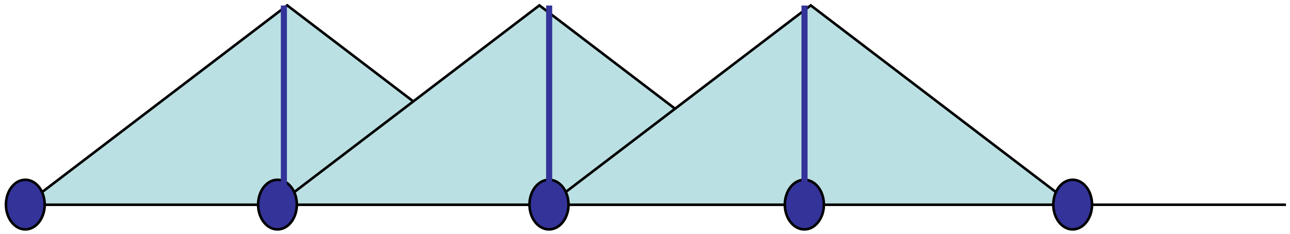
- **Then we calculate the derivatives and get an equation system**

$$\Pi = \Pi(a_1, a_2, ..., a_n)$$

$$\frac{d\Pi}{da_i} = equation(a_1, a_2, ..., a_n)$$

Base of Finite Elements

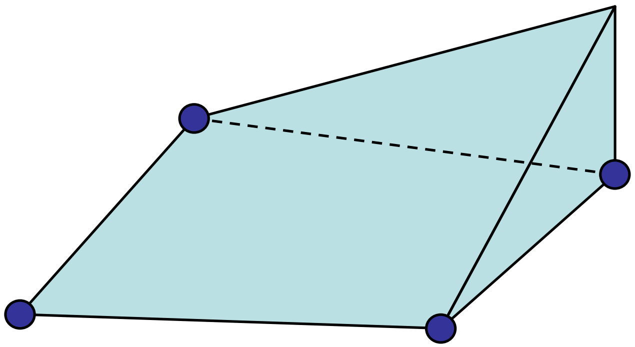
- **Select special simple trial functions with only a local influence !**
- **Example along an edge**

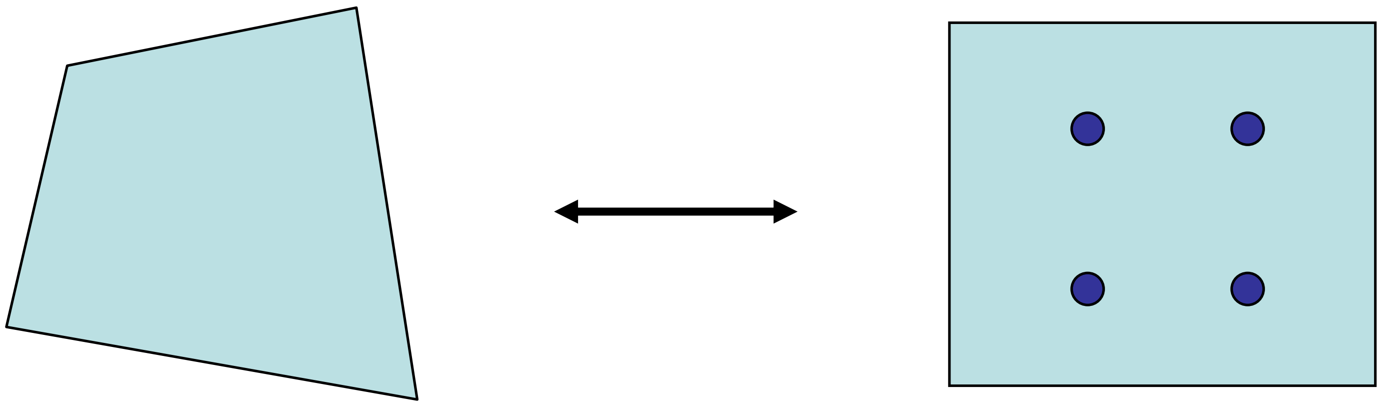


- **Local triangular (linear) functions**
  Sum of functions is 1.0 everywhere

Discretisation of an area

- Triangles
- Quadrilaterals





Real Geometry

Unit square
with 4 Gauss-Points

Element matrices

- Linear Functions
  => constant derivatives
- Although the selected functions to not fulfil the DE, the solution may be obtained by the variational principle and numerical integration
- Each „node“ is one unknown
  and we might have millions of them
- „Elements“ have only local influence
- Large equation systems have sparse matrices allowing for fast solution.

Discretisation in Time

- **Assumption:**

$$\frac{\partial T}{\partial t} = \frac{T_i - T_{i-1}}{\Delta t}$$

- **Establishing the Equations for time $t_{i-1}+\Theta\Delta t$**

$$\left(\Theta \cdot A + \frac{1}{\Delta t} C \right) T_i = \left((\Theta - 1) \cdot A + \frac{1}{\Delta t} C \right) T_{i-1} - Q$$

$$A = \int_{\Omega} k_{ij} \frac{\partial N}{\partial x_i} \frac{\partial N}{\partial x_j} d\Omega \; ; \; C = \int_{\Omega} S \left(N^T N \right) d\Omega$$

Stability of Method

- **The parameter Θ of the Crank-Nicholson Method may be selected between 0.5 and 2.0 to obtain stable Methods. The default is 0.7**
- **The time step has to be selected to allow the physical effects to be traced properly. An adaptive selection may be needed for many cases.**
- **For small time steps the matrix C will become dominant. For a certain critical value of Δt the sum of $a_{ij} + c_{ij}/\Delta t$ will change the sign. If this happens the discrete maximum principle is violated and we will get perceptable oscillations in our solution. Remedy: use lumped (diagonal) matrices for C**

Heat Flow Material constants

- T = Temperature [° K]
- u = Vector of heat fluxes [W/m^2]
- q = Power of sources or sinks [W/m^3]
- S = $\rho \cdot c_p$ = Storage coefficient in [Wsec/K·m^3]
  where
  ρ = specific weight of the material [kg/m^3]
  c_p = specific thermal capacity [Wsec/K·kg]

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Heat Flow
Boundary Conditions

- **Prescribed temperature**

$$T = T_0 \ [^\circ \ ]$$

- **Prescribed flux**

$$q = q_0 \ [W/m^2]$$

- **Prescribed resistance to environment temperature**

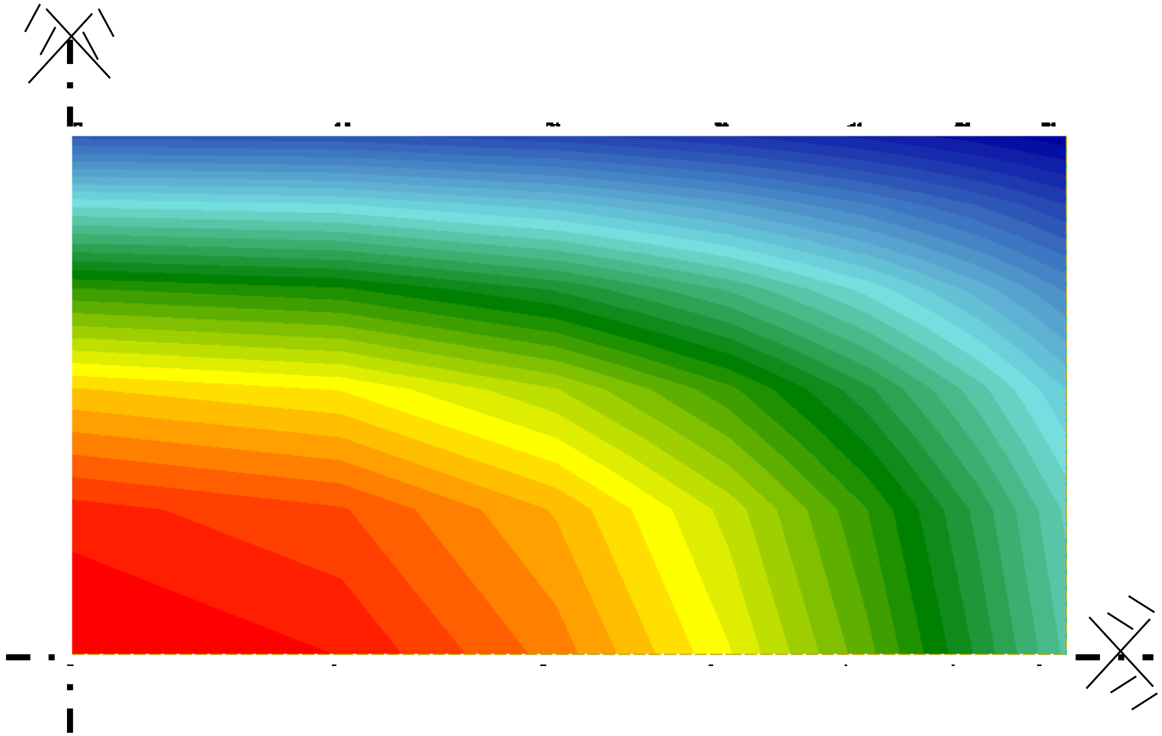
$$q = \alpha \ (T - T_0)$$

- **Radiation**

$$q = \varepsilon \ k \ (T^4 - T_0^4)$$

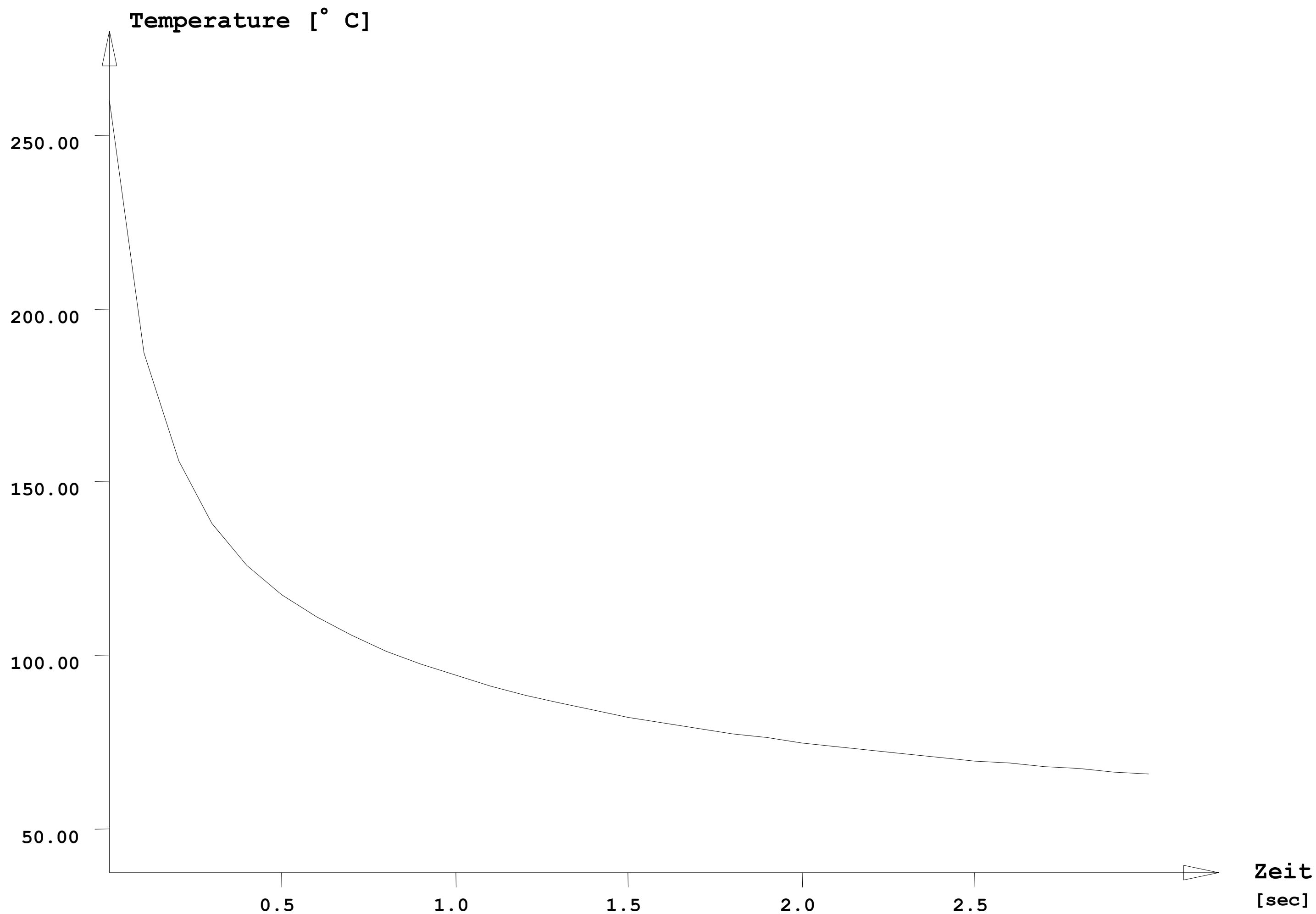
ε = Emission grade
k = Boltzmann constant

Example

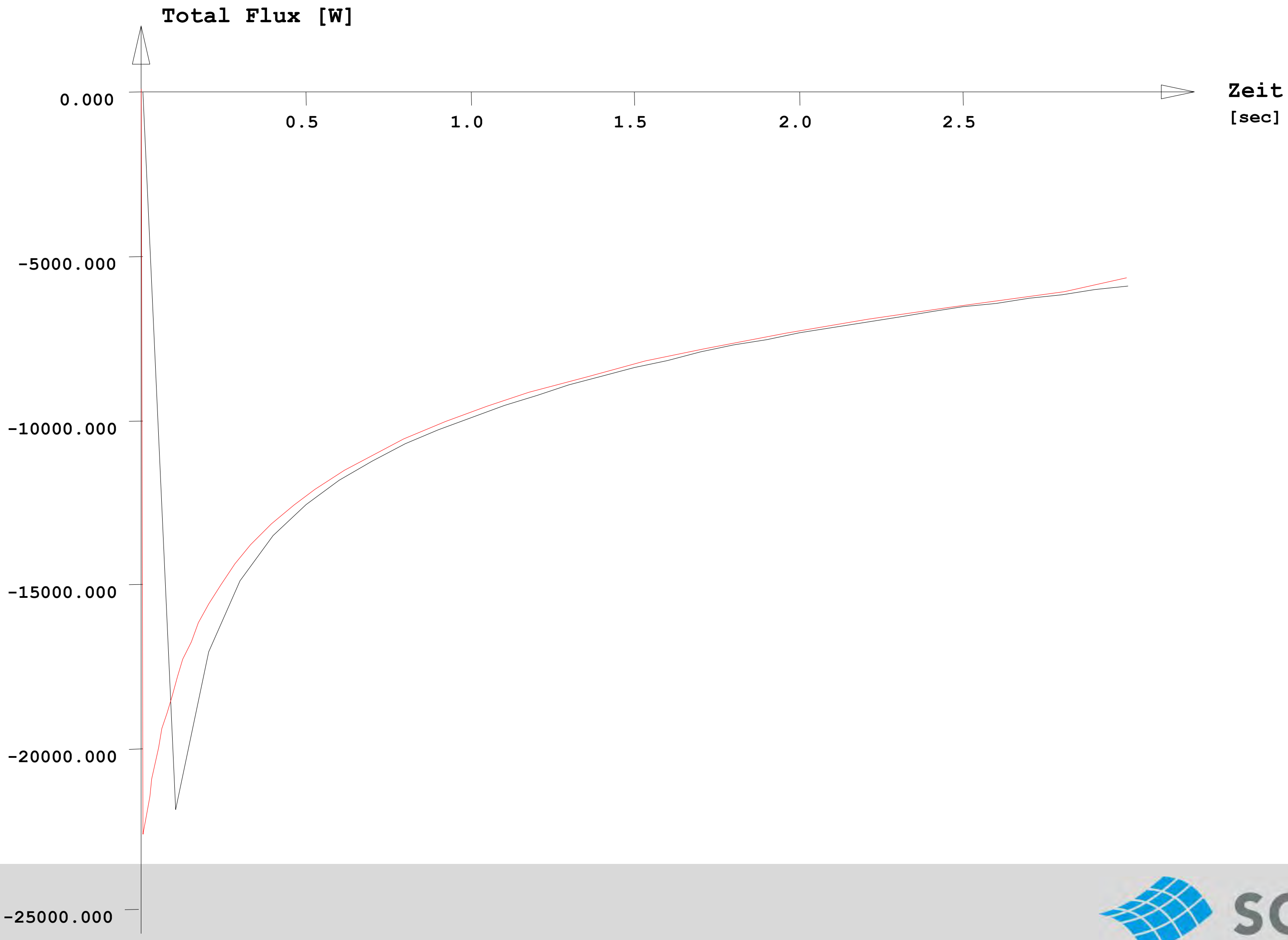


- A hot block with 260 degrees is thrown in a isothermal oil with 37 degrees.
- A sudden heat flow is created at the surface, propagating into the internal area, reducing within time.

Temperature



Heat Flux



Total Flux [W]
0.000
-5000.000
-10000.000
-15000.000
-20000.000
-25000.000
0.5
1.0
1.5
2.0
2.5
Zeit
[sec]

Hydration of Concrete

- **During the hydration of concrete heat is produced. The production of heat is dependant on the heat.**
- **The thermal heating and cooling creates stresses**
- **The tensile strength is developing with the time**
- **The sensibility of the required reinforcement is highly dependant on the parameters of that process.**
- **First problem is to find basic rules behind that process**
- **Second problem is to convert those rules to software**
- **And the last problem is to get the real material data**

Basics

- **Base for all properties, strength, elasticity modulus and heat production is the hydration grade α, defined as the ratio of the heat already produced to the maximum heat possible an the end of time:**

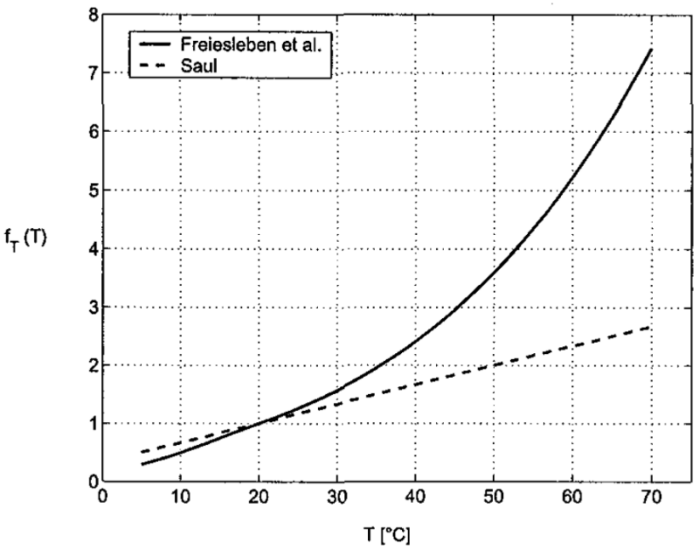
$$\alpha = \frac{Q(\tau_w)}{Q(\infty)}$$

The heat production per time is then given by the derivative of that function

Effective Age of Concrete

- **Empirical (Saul etc.)**

$$\tau_w = \int \left[\frac{T + T_{ref}}{20 + T_{ref}} \right]^S dt$$



- **Theoretical (Freiesleben etc.)**

$$\tau_w = \int \exp\left(\frac{A}{R} \left(\frac{1}{293} - \frac{1}{273 + T} \right) \right) dt$$

$$R = 8.3143 \left[J / mol\, K \right] ; \; A = 33500 + 1470 \cdot \max(0, 20 - T)$$

Hydration Relations

- **Jonasson**

$$\alpha = \exp\left[b \cdot \ln\left(1 + \frac{\tau_w}{\tau_k}\right)^a \right]$$

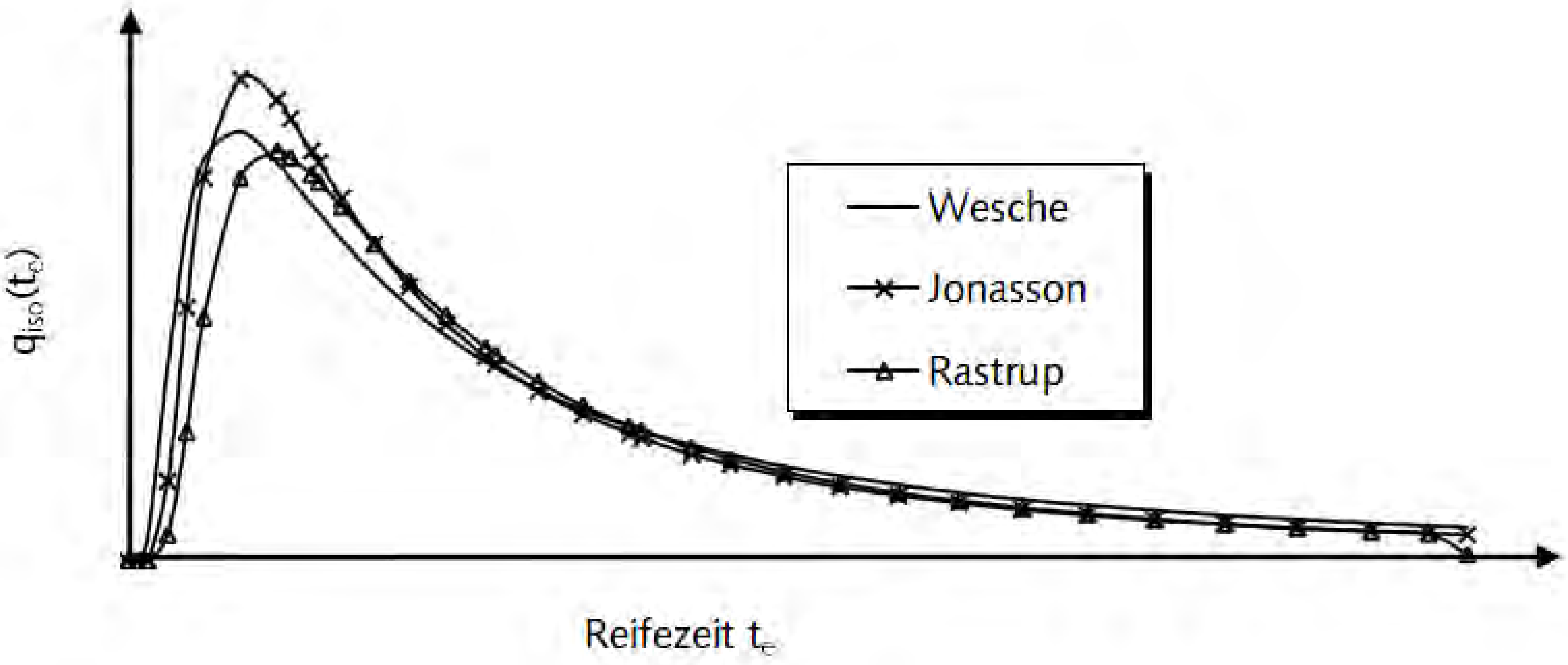
- **Shrinkage Core**

$$\alpha = \frac{a \cdot (\tau_w - \tau_k)}{1 + a \cdot (\tau_w - \tau_k)}$$

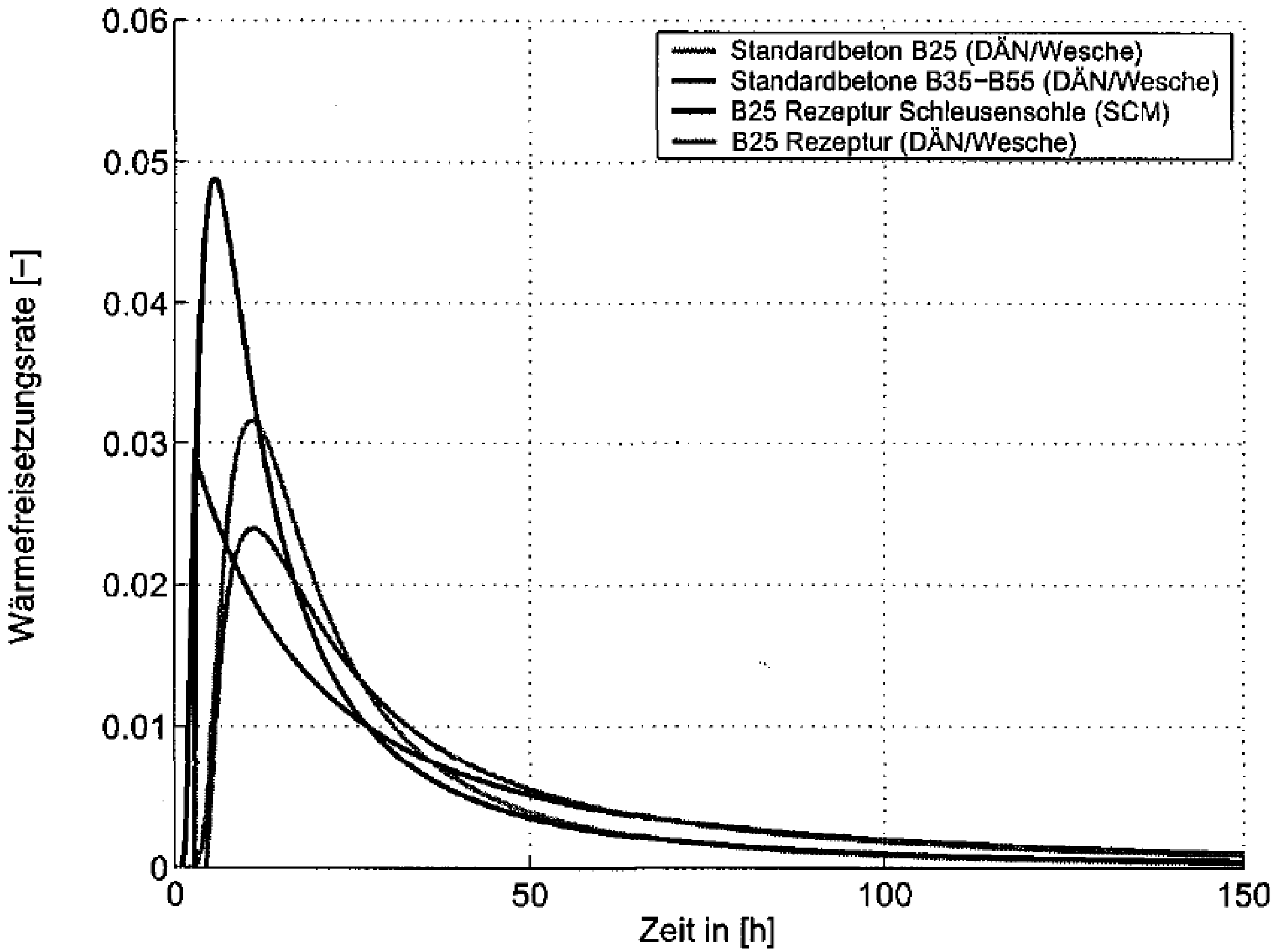
- **Wesche**

$$\alpha = \exp\left[-\left(\frac{\tau_w}{\tau_k}\right)^b \right]$$

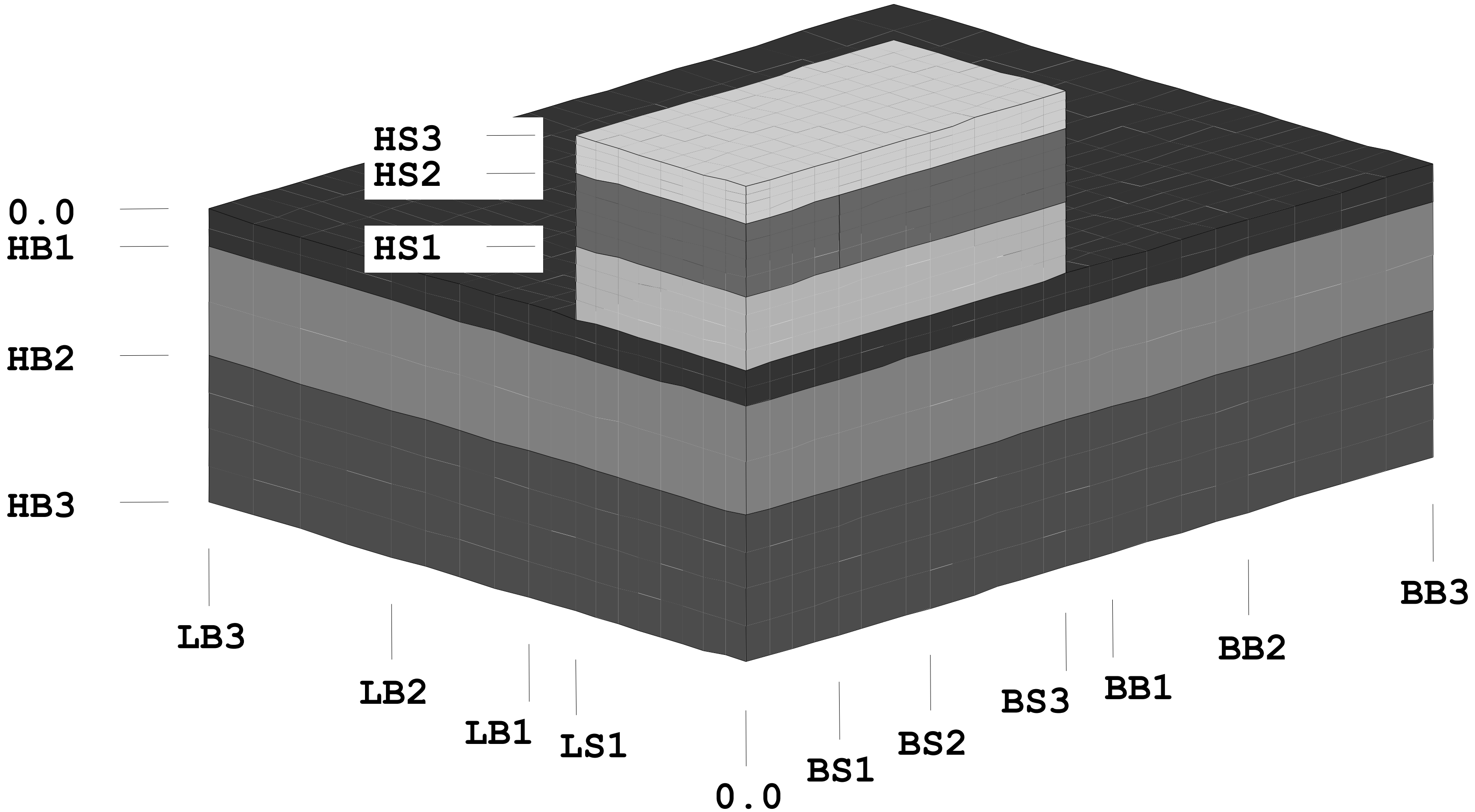
Hydration Relations



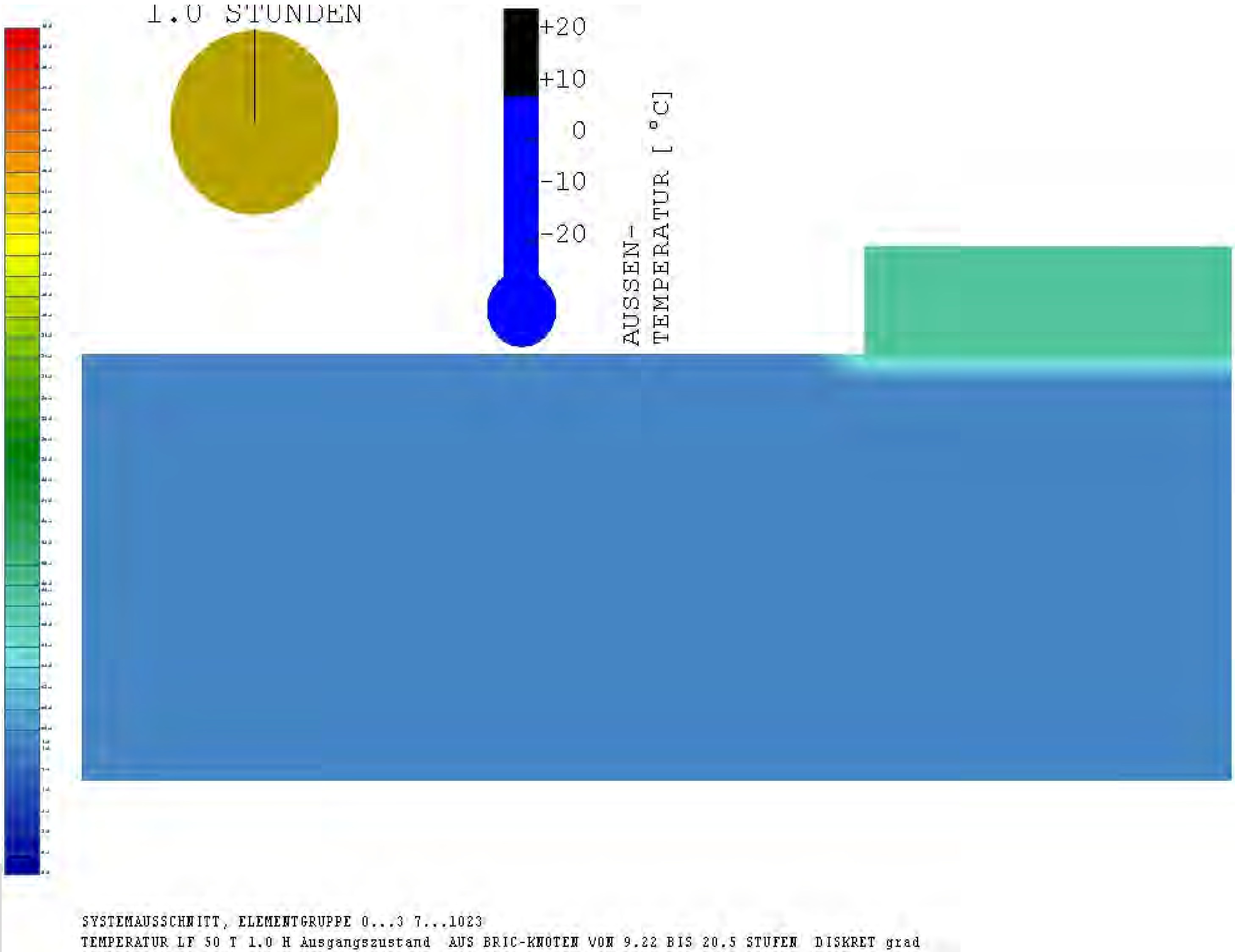
Hydration Relations

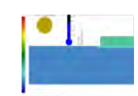


Example Hydration



Temperatures of Hydration





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Example: Fire Design

- **Materials, especially steel looses its strength when exposed to fire.**
- **So steel will be covered/combined with concrete to protect it.**
- **To design a column we need the heat distribution within the section after 90 min (Fire class R90)**
- **From that we may calculate the hot ultimate bearing capacity**

Parameters for Analysis

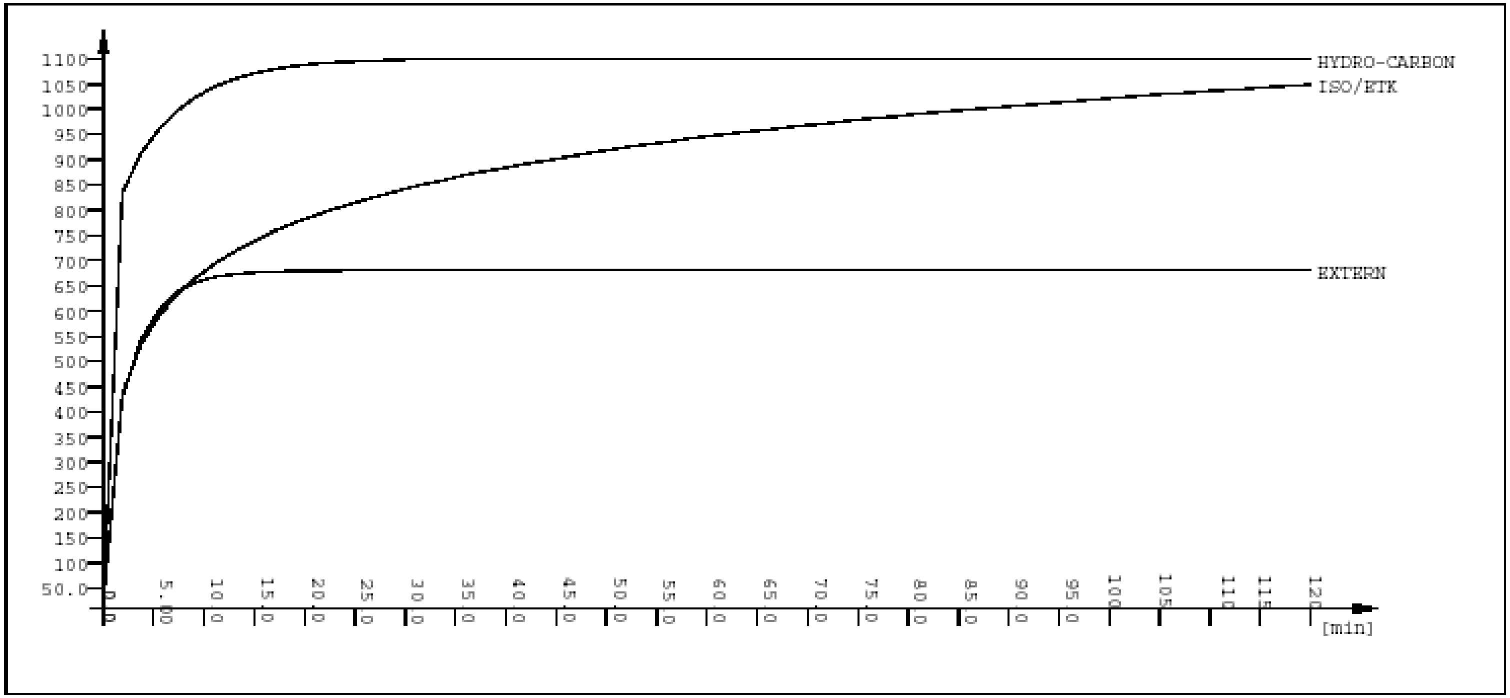
- Transient analysis
- Standard fire design temperature curve T=T(t)
- Radiation boundary conditions are dominant
- Temperature dependant conductivity
- Temperature dependant capacity including evaporation of pore water in concrete at 100 degree

Temperature Curves

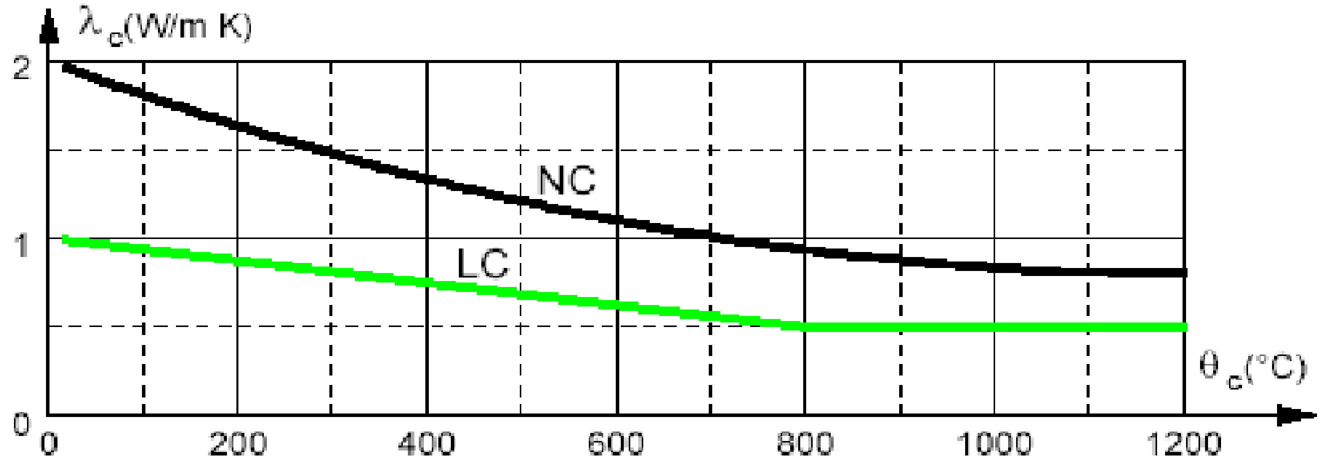
$$T = 20 + 345\log_{10}(8t + 1) \tag{53}$$

$$T = 20 + 660\cdot(1 - 0.687\cdot e^{-0.32t} - 0.313\cdot e^{-3.8t}) \tag{54}$$

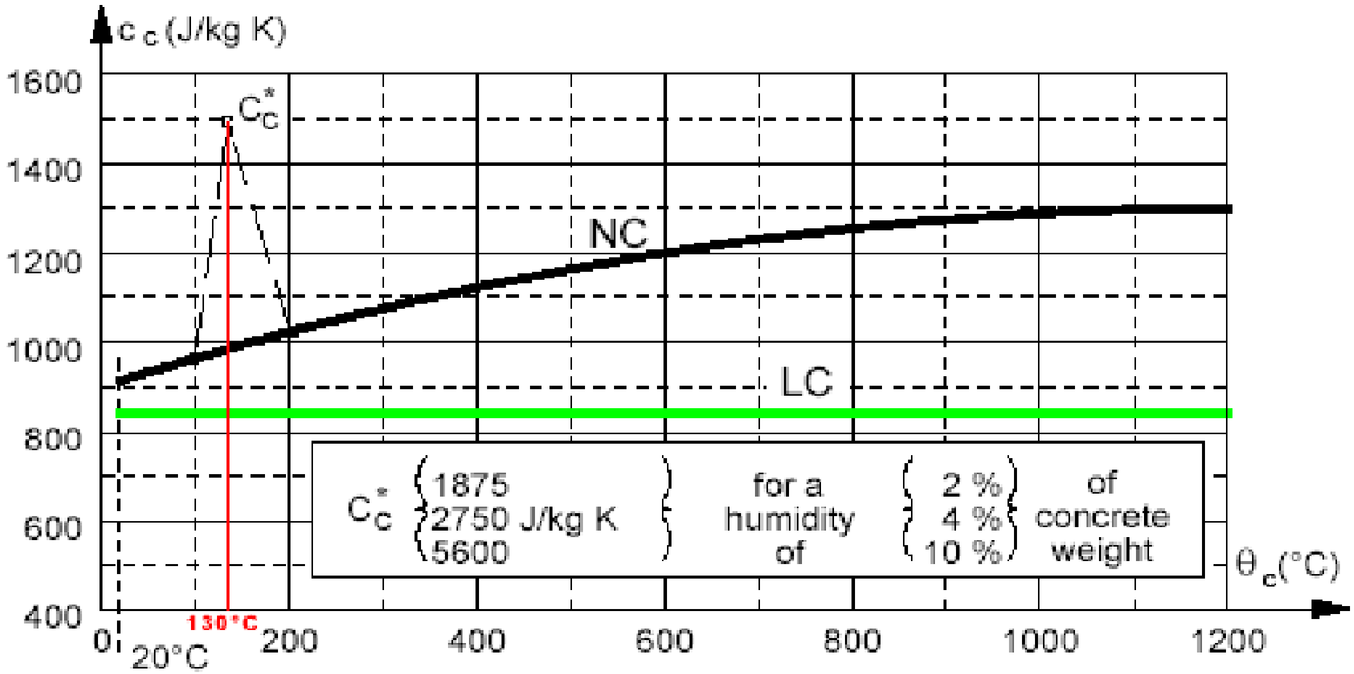
$$T = 20 + 1080\cdot(1 - 0.325\cdot e^{-0.167t} - 0.675\cdot e^{-2.5t}) \tag{55}$$



Concrete

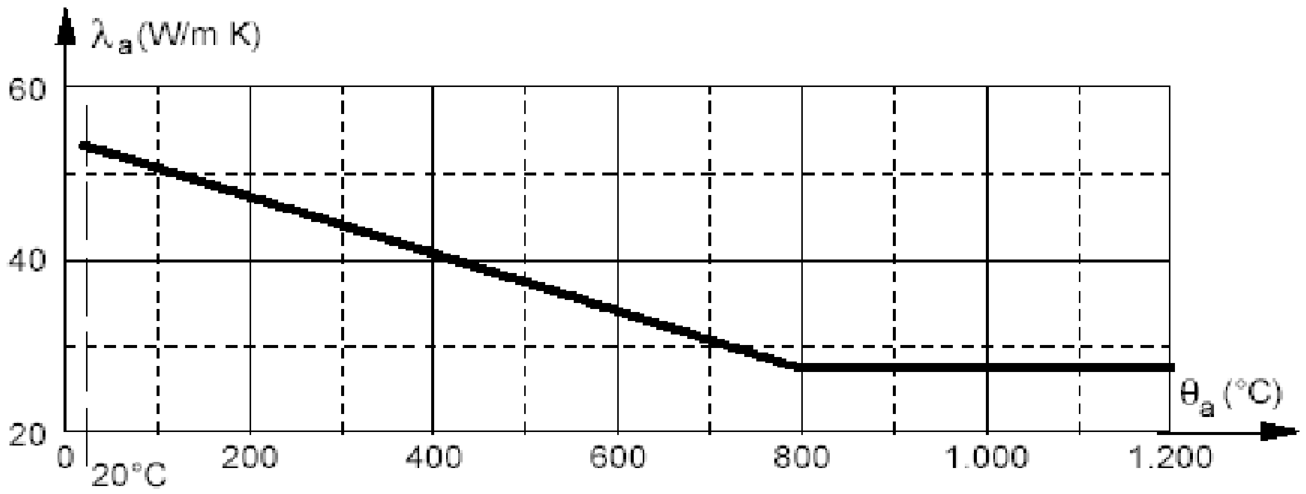


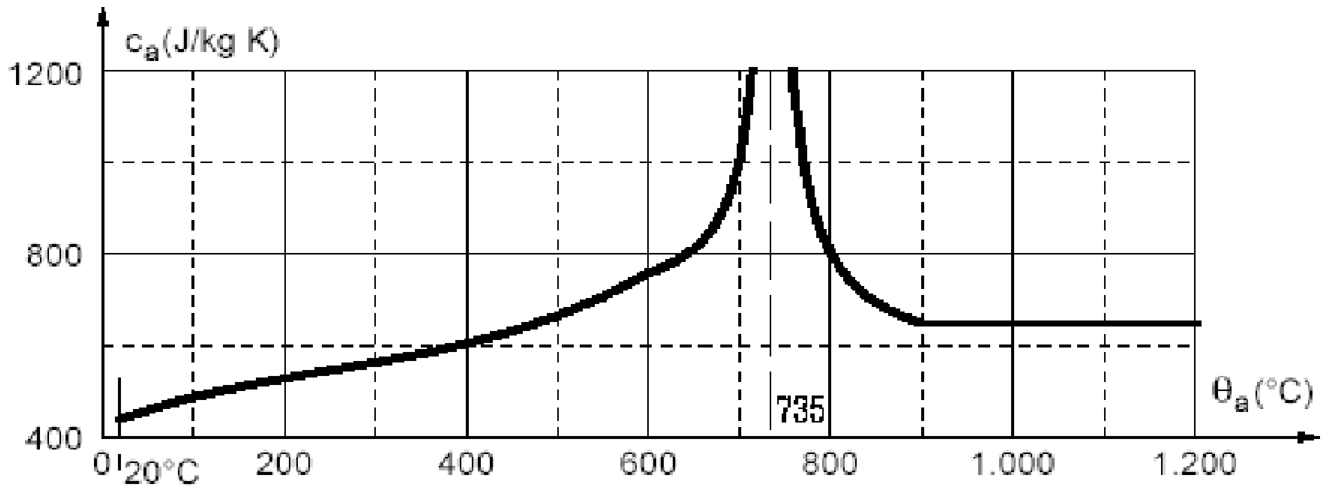
λ_c(W/m K)
2
1
0
NC
LC
θ_c(°C)
0
200
400
600
800
1000
1200

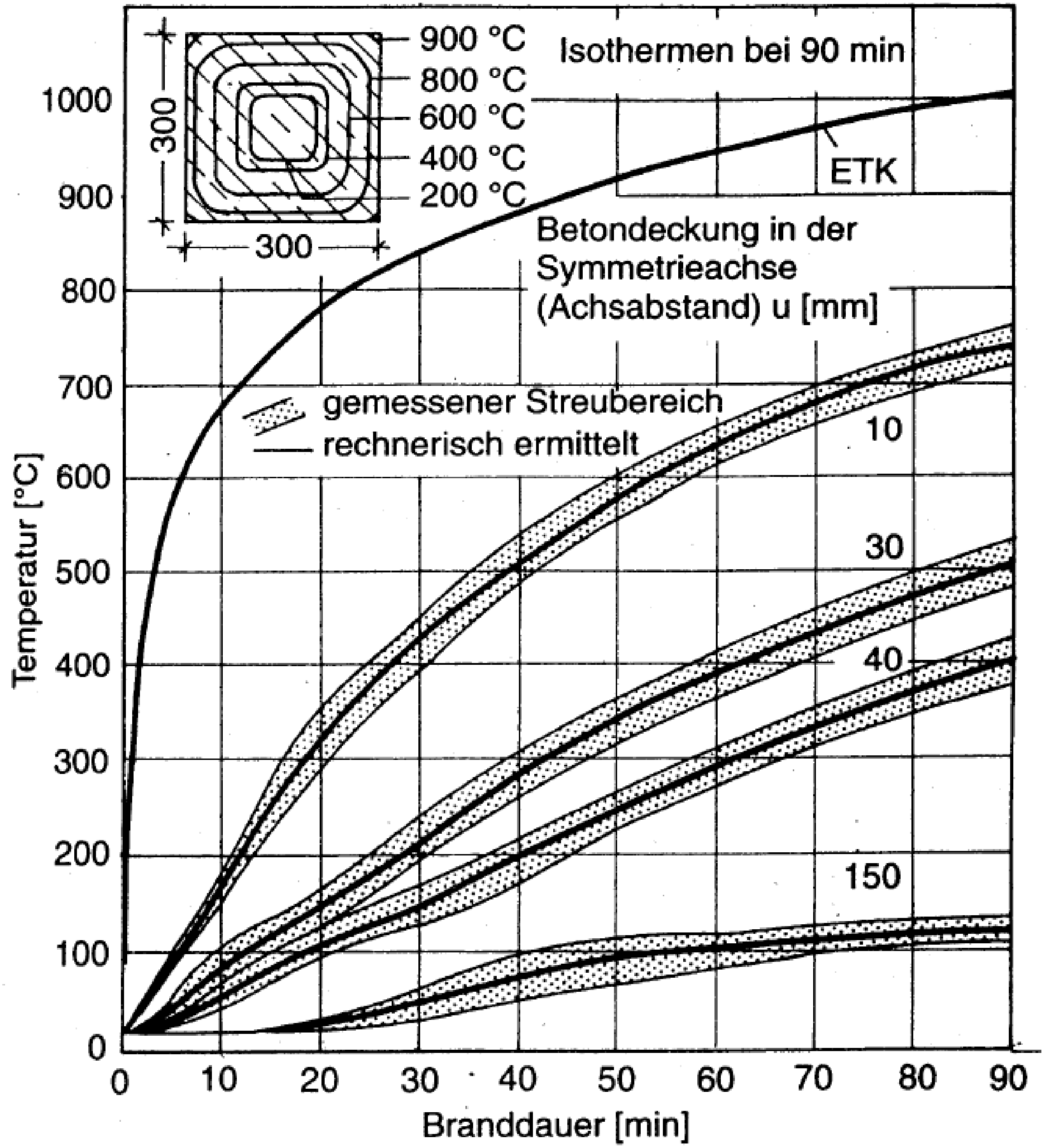


c_c (J/kg K)
1600
1400
1200
1000
800
600
400
C_c^*
NC
LC
C_c^* { 1875 / 2750 J/kg K / 5600 } for a humidity of { 2 % / 4 % / 10 % } of concrete weight
θ_c(°C)
130°C
0
20°C
200
400
600
800
1000
1200

Steel

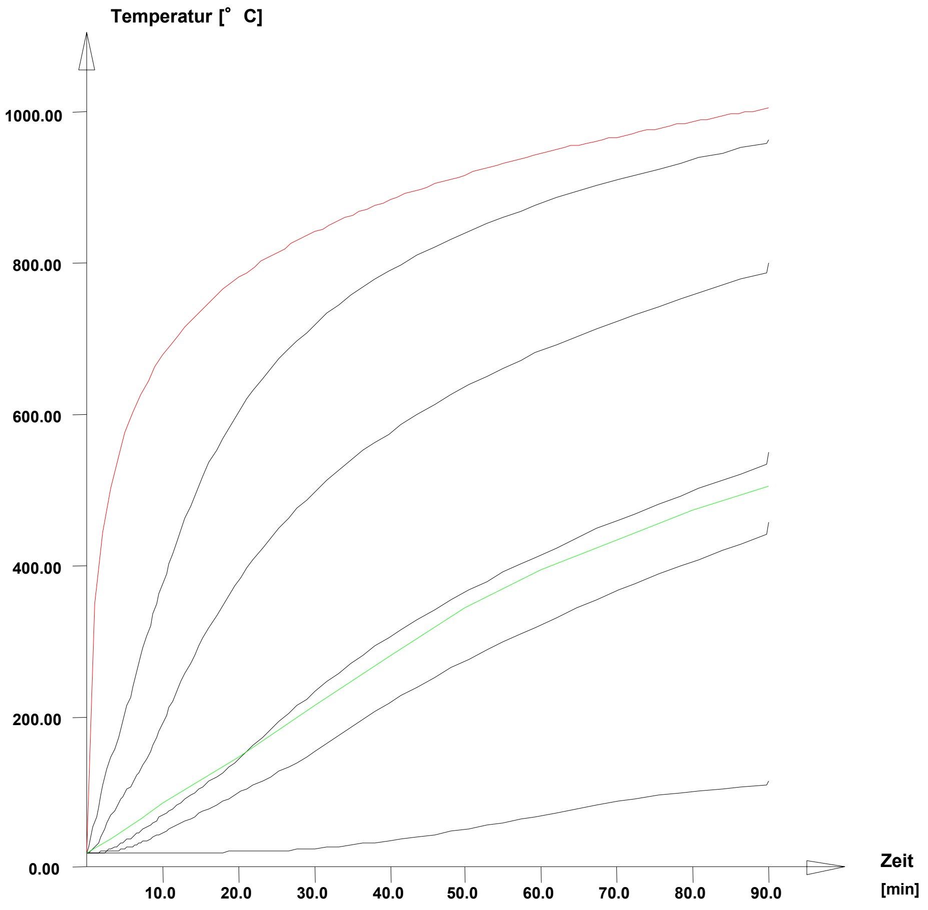

λa(W/m K)
60
40
20
0
20°C
200
400
600
800
1.000
1.200
θa (°C)

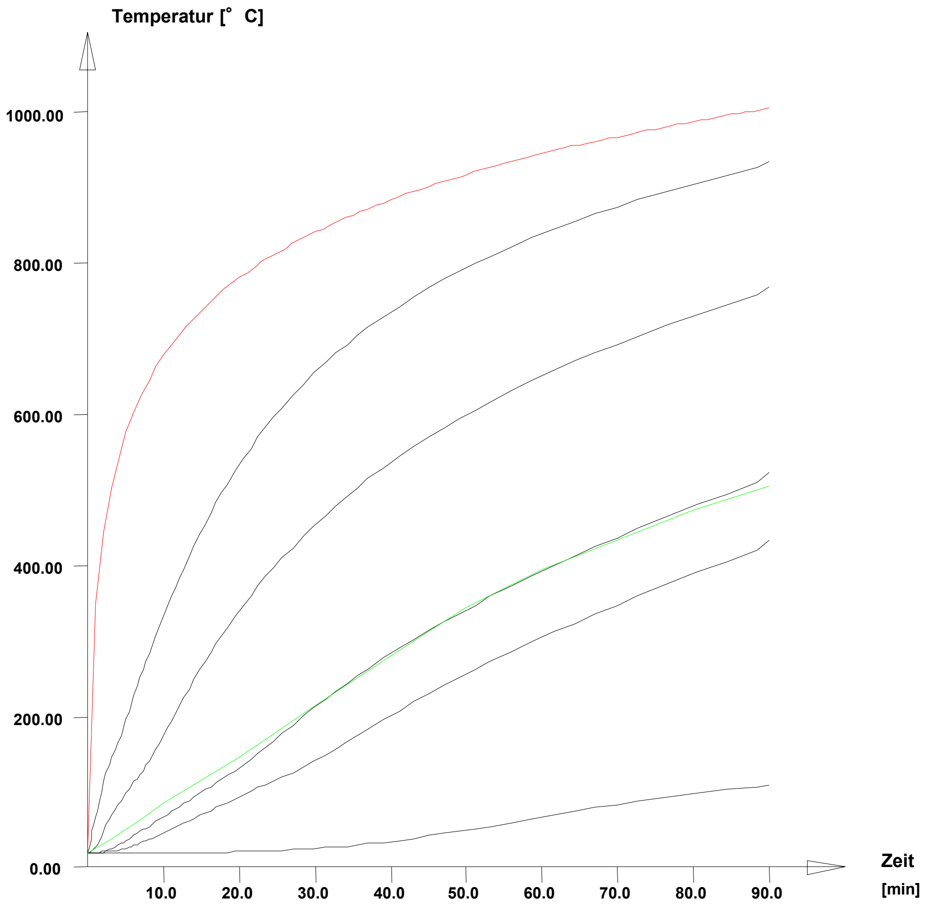

ca(J/kg K)
1200
800
400
0
20°C
200
400
600
800
1.000
1.200
735
θa(°C)



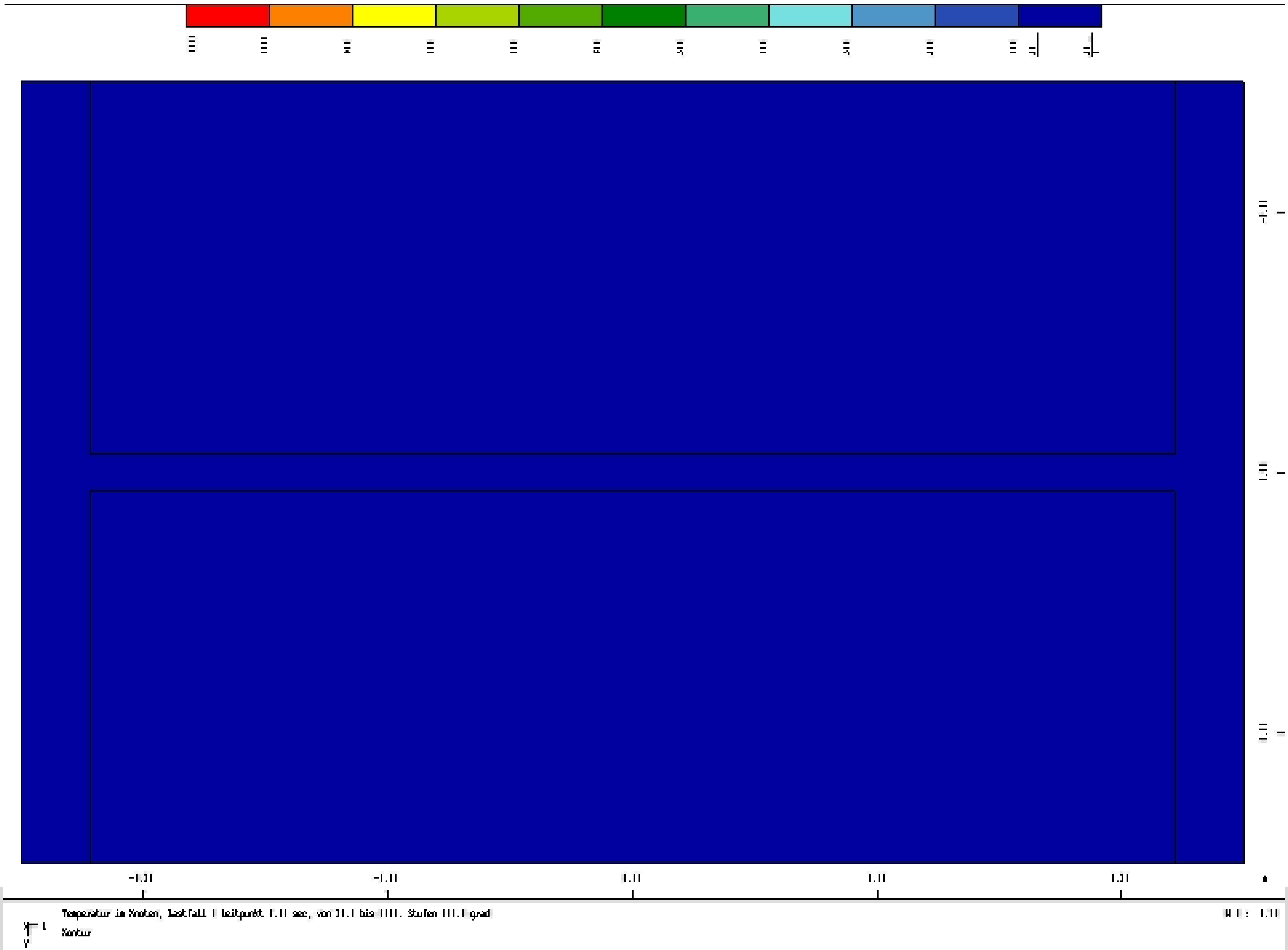
900 °C
800 °C
600 °C
400 °C
200 °C
300
300
Isothermen bei 90 min
ETK
Betondeckung in der Symmetrieachse (Achsabstand) u [mm]
gemessener Streubereich
rechnerisch ermittelt
10
30
40
150
Temperatur [°C]
Branddauer [min]
0
100
200
300
400
500
600
700
800
900
1000
10
20
30
40
50
60
70
80
90

Analysis Results

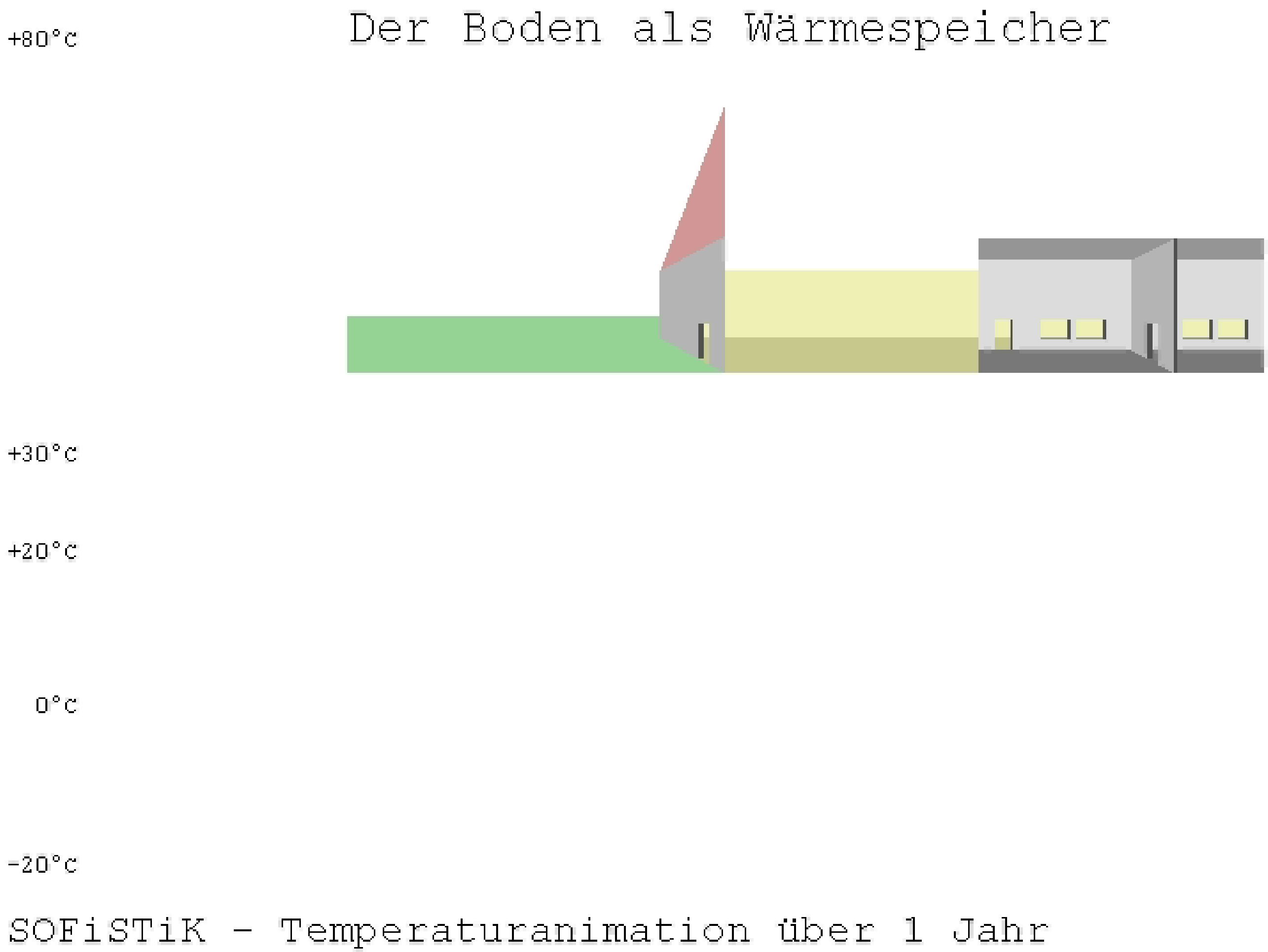




FE for composite Column



Example: Heat Storage System



Further fields of application

- **Groundwater seepage**
- **Moisture transport**
- **Electric Fields**
- **Magnetic fields**
- **Shear stress in bars**
- **Lubrication problems**