Series system reliability of uncertain linear structures under Gaussian excitation by cross entropy-based importance sampling

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ABSTRACT

We present an adaptive importance sampling (IS) method to estimate the reliability of linear structures with parameter uncertainties that are subjected to Gaussian process excitation. Structural failure is defined as a union of multiple first-passage failure events. The main contribution is the introduction of an efficient IS density for the uncertain structural parameters. This density is determined by minimizing the cross entropy (CE) between the theoretically optimal IS density of the structural parameters and a chosen parametric family of probability distributions. The CE minimization procedure requires evaluation of the system failure probability conditional on fixed values of the uncertain parameters. A closed-form analytical approximation of this conditional failure probability is derived based on an upper bound on the out-crossing rate. Finally, a joint IS density of the random excitation and the uncertain structural parameters is introduced to estimate the series system failure probability involving parameter uncertainties. The accuracy and efficiency of the proposed method is demonstrated by means of two examples: the first involves a Gaussian white noise-excited two-story linear shear frame and the second involves a six-story three-bay moment resisting steel frame subjected to a filtered non-stationary Gaussian excitation.

INTRODUCTION

Structural reliability analysis aims at computing the probability of failure of a structure with respect to a prescribed failure criterion by accounting for the uncertainties in the structural parameters (the geometric and material properties) and the external loading. When the load is dynamic, such as the one arising from earthquakes, wind or sea waves, the reliability is estimated in terms of the first-passage probability, i.e., the probability that the dynamic response of the structure exceeds a prescribed threshold level over the duration of the excitation. In general, reliability analysis is usually classified into two categories: component reliability, which considers only a single mode of failure, and system reliability, in which multiple failure modes are considered. This paper focuses on the estimation of series system reliability of uncertain linear structures subjected to Gaussian process excitation. Here the system failure event is defined as the union of first-passage events associated with multiple critical responses.

The series system reliability cannot be directly deduced from the marginal first-passage probabilities of the output responses if the component failure events are statistically dependent. Such dependence is usually present when the component first-passage events occur due a common source of excitation or when the resistances of the components are dependent. For the case where the structural parameters are deterministic and the applied excitation is modeled as a Gaussian process, there are several approaches to estimate the series system reliability. Analytical approximations of the failure probability based on the joint out-crossing rate of Gaussian responses processes are proposed in (Li and Melchers 1993; Song and Der Kiureghian 2006). Bounds can be obtained on the system reliability using analytical bounding formulas (Melchers and Beck 2018) or linear programming (Song and Der Kiureghian 2003; Byun and Song 2020). Alternatively, the Monte Carlo simulation (MCS) method can be applied to estimate the system failure probability. This approach is devoid of assumptions (Poisson assumption for the number of out-crossings) and, hence, is asymptotically exact with increase in the number of samples. The main challenge in applying MCS lies in controlling the sampling variance of the failure probability estimator; the aim is to obtain probability estimates of acceptable accuracy with a small number of dynamic model runs. Reduction in sampling variance is achieved by advanced Monte Carlo methods such as subset simulation (Au and Beck 2001a), Girsanov's transformation-based IS (Kanjilal and Manohar 2019) and line sampling (Koutsourelakis et al. 2004; Schuëller et al. 2004b). Efficient simulation methods that are specific to deterministic linear dynamical systems have also been developed. These methods increase the efficiency of estimation by utilizing the linearity of the structural response with respect to the Gaussian loading. The central theme of these methods is to express the failure region of the series system in terms of a large number of linear failure regions corresponding to the failure of a particular output response at a particular time instant. In (Au and Beck 2001b) this strategy is applied to design an effective IS density of the random excitation. Other approaches to estimate the system reliability based on this concept are studied in (Katafygiotis and Cheung 2004; Katafygiotis and Cheung 2006; Misraji et al. 2020).

When the structural parameters are uncertain and the excitation is a Gaussian random process, the response is non-linear with respect to the structural parameters. In this case, estimation of the series system reliability becomes considerably more involved. The MCS method is the most viable approach to tackle this class of problems. The subset simulation method can be readily applied to this case. Alternatively, efforts to extend the tailored approaches for Gaussian process-excited deterministic linear systems to deal with structural parameter uncertainties have been attempted in (Jensen and Valdebenito 2007; Pradlwarter and Schuëller 2010; Valdebenito et al. 2014). In these studies, the system failure probability conditional on a given realization of the uncertain parameters is determined using the approach in (Au and Beck 2001b). The unconditional failure probability of the series system is then estimated by integrating the conditional probability over the domain of the uncertain parameters by importance sampling (Jensen and Valdebenito 2007; Valdebenito et al. 2014) or line sampling (Pradlwarter and Schuëller 2010). These methods require system specific information to facilitate reliability estimation. In (Jensen and Valdebenito 2007; Pradlwarter and Schuëller 2010), a pseudo-design point with respect to the uncertain structural parameters has to be identified. These approaches can be effective when there is a unique design point (in the parameter space) contributing to the failure probability. The IS method in (Valdebenito et al. 2014) makes use of a surrogate model for the probability of failure as a function of the uncertain parameters. The performance of the method thus relies on the proper choice of the surrogate model, which is not a straightforward task when the number of uncertain parameters is large or the dependence of the conditional failure probability on the parameters is strongly non-linear.

The present contribution develops an adaptive importance sampling method to estimate the series system reliability of uncertain linear structures subjected to Gaussian loading. It is an extension of a recently developed method for component-level first-passage probability estimation of structures with parameter uncertainty (Kanjilal et al. 2021). The proposed approach employs the strategy presented in (Au and Beck 2001b) to construct a conditional (on a fixed value of the structural parameters) IS density of the random loading. A novel IS density of the uncertain structural parameters is then introduced. This IS density is obtained through application of the cross entropy (CE) method. The CE method is an adaptive sampling approach that determines a near-optimal IS density through minimizing the Kullback-Leibler (KL) divergence between the theoretically optimal IS density and a chosen parametric family of probability distributions. We discuss appropriate distribution models for the chosen parametric family, depending on the dimension of parameter uncertainties and the number of failure modes. The CE optimization requires evaluation of the system failure probability conditional on sample realizations of the uncertain parameters. To ensure smooth convergence to the optimal IS density, we employ an analytical approximation of the conditional failure probability during optimization. The approximation is based on an upper bound on the joint out-crossing rate of the critical responses (Li and Melchers 1993). In this study, we derive a closed-form analytical solution of the upper bound, which enables faster evaluation of the conditional probability during CE optimization. Finally, a joint IS density of the uncertain structures and the random excitation is considered to estimate the failure probability of the series system. Unlike the methods in (Jensen and Valdebenito 2007; Pradlwarter and Schuëller 2010; Valdebenito et al. 2014), the proposed approach is completely adaptive and can be used as a black-box algorithm as it does not require problem-specific adjustments. It is therefore more robust and generally applicable to any linear dynamical system.

PROBLEM FORMULATION

Linear Dynamical System

We consider an n degree-of-freedom linear structure with uncertain parameters subjected to non-stationary stochastic excitation. The governing equation describing the response of the structure is expressed as

$$\mathbf{M}(\boldsymbol{\Theta})\ddot{\boldsymbol{X}}(t) + \mathbf{C}(\boldsymbol{\Theta})\dot{\boldsymbol{X}}(t) + \mathbf{K}(\boldsymbol{\Theta})\boldsymbol{X}(t) = \mathbf{D}\boldsymbol{f}(t), \tag{1}$$

where $\ddot{\boldsymbol{X}}$, $\dot{\boldsymbol{X}}$ and $\boldsymbol{X}$ are the $n \times 1$ acceleration, velocity and displacement vectors, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $n \times n$ mass, damping and stiffness matrices, $\boldsymbol{\Theta}$ is an $n_{\boldsymbol{\theta}} \times 1$ vector of random variables that model the uncertain structural parameters, $\boldsymbol{f}$ is an $l \times 1$ vector of random dynamic loads acting on the structure over a time span $t \in [0, T]$ and $\mathbf{D}$ is an $n \times l$ matrix that couples the external excitation with the degrees of freedom of the structure. We consider the case where the components of $\boldsymbol{f}$ are Gaussian random processes.

Let $\{Z_i; i = 1, \ldots, m\}$ be m critical output responses. In a linear system, the relationship between the input excitation and the output response is linear, and can be written as

$$Z_i(t, \boldsymbol{\theta}) = \sum_{j=1}^{l} \int_0^t K_{ij}(t-\tau; \boldsymbol{\theta}) f_j(\tau) d\tau = \int_0^t \boldsymbol{K}_i^{\mathrm{T}}(t-\tau; \boldsymbol{\theta}) \boldsymbol{f}(\tau) d\tau. \tag{2}$$

In the above equation, $\boldsymbol{\theta}$ denotes a particular outcome of the uncertain structural parameter vector $\boldsymbol{\Theta}$ and $K_{ij}(t; \boldsymbol{\theta})$ denotes the unit impulse response function for the i-th output at time t due to a unit impulse applied at the j-th input at time $t = 0$, where the outputs are assumed to start from zero initial conditions without loss of generality. Consider a discrete time representation of the time interval $[0, T]$. Let $\{t_k = (k-1)\Delta t; k = 1, \ldots, n_T\}$ be the time instants of analysis, where n_T is the number of time points and $\Delta t = T/(n_T - 1)$ is the time step size. Let $\boldsymbol{f}(t_k)$ denote the stochastic excitation at time $t = t_k$. For Gaussian loading, one can represent $\boldsymbol{f}(t_k)$ by a linear combination of an $n_{\xi} \times 1$ standard Gaussian random vector $\boldsymbol{\Xi}$ as $\boldsymbol{f}(t_k) = \mathbf{G}_k \boldsymbol{\Xi}$, where $\{\mathbf{G}_k, k = 1, \ldots, n_T\}$ are appropriate deterministic matrices. Then the discrete-time analog of the input-output relationship in Eq. (2) is given by

$$Z_i(t_k, \boldsymbol{\theta}, \boldsymbol{\Xi}) = \sum_{s=1}^{k} c_s \boldsymbol{K}_i^{\mathrm{T}}(t_k - t_s; \boldsymbol{\theta}) \boldsymbol{f}(t_s) \Delta t = \boldsymbol{r}_{i,k}^{\mathrm{T}}(\boldsymbol{\theta}) \boldsymbol{\Xi}, \tag{3}$$

where $\boldsymbol{r}_{i,k}^{\mathrm{T}}(\boldsymbol{\theta}) = \sum_{s=1}^{k} c_s \boldsymbol{K}_i^{\mathrm{T}}(t_k - t_s; \boldsymbol{\theta}) \mathbf{G}_s \Delta t$ and c_s is a coefficient that depends on the particular numerical integration scheme used to integrate Eq. (2).

Series System Reliability

Reliability analysis of dynamical systems involves the computation of the first-passage probability. In a series system defined in terms of m output responses, first-passage failure occurs when any one of the outputs $\{Z_i, i = 1, \ldots, m\}$ exceeds a corresponding threshold level z_i^* within the time duration T. The system level failure event F is therefore expressed as

$$F = \bigcup_{i=1}^{m} F_i, \tag{4}$$

where

$$F_i = \left\{ \boldsymbol{\theta} \in \mathbb{R}^{n_\theta}, \boldsymbol{\xi} \in \mathbb{R}^{n_\xi} : \max_{k=1,\ldots,n_T} |Z_i(t_k, \boldsymbol{\theta}, \boldsymbol{\xi})| \geq z_i^* \right\} \tag{5}$$

denotes first-passage failure with respect to the i-th output response. The probability of occurrence of F can be expressed by means of the multi-dimensional integral

$$P_F = \int_{\boldsymbol{\theta} \in \mathbb{R}^{n_\theta}} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}, \tag{6}$$

where

$$P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \int_{\boldsymbol{\xi} \in \mathbb{R}^{n_\xi}} \mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F\} p_{\boldsymbol{\Xi}}(\boldsymbol{\xi}) d\boldsymbol{\xi} \tag{7}$$

represents the conditional probability of failure of the system given the uncertain parameters $\boldsymbol{\theta}$. In the above equations, $p_{\boldsymbol{\Xi}}(\boldsymbol{\xi})$ and $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, respectively, denote the joint probability density function (PDF) of $\boldsymbol{\Xi}$ and $\boldsymbol{\Theta}$, and $\mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F\}$ is the indicator function for the failure event which takes the value 1 if $(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F$ and is 0 otherwise.

A convenient way to evaluate P_F is by Monte Carlo simulation (MCS). In principle, one could use the standard Monte Carlo method. When the probability of failure is small, standard MCS requires a very large number of dynamical system evaluations to generate accurate results. In this paper we develop an alternative strategy based on importance sampling (IS) to estimate the series system reliability. To this end, we note that the study in (Au and Beck 2001b) introduces an IS density of the random vector $\boldsymbol{\Xi}$ modeling the dynamic load, which enables efficient estimation of the conditional probability of failure $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ by IS. Therefore, the key challenge in the construction of an efficient IS estimator for P_F lies in the design of an effective IS density to integrate $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ over the domain of the uncertain parameter vector $\boldsymbol{\Theta}$. In the next section, we present a novel approach to determine this IS density using the cross entropy method. Subsequently, we combine the proposed IS density of $\boldsymbol{\Theta}$ with the IS density of $\boldsymbol{\Xi}$ to obtain the estimator for the first-passage probability of the series system.

DETERMINATION OF THE IS DENSITY OF $\boldsymbol{\Theta}$

Evaluation of the dynamic system reliability of uncertain structures according to Eq. (6) requires integration of the conditional failure probability $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ over the sample space of $\boldsymbol{\Theta}$. In order to evaluate the integral by IS, an IS density $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ of the structural parameters is introduced. The integral in Eq. (6) is modified to

$$P_F = \int_{\boldsymbol{\theta} \in \mathbb{R}^{n_\theta}} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) W(\boldsymbol{\theta}) h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}, \tag{8}$$

where $W(\boldsymbol{\theta}) = p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) / h_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is the importance weight function. Based on Eq. (8), one can estimate P_F using the following IS estimator:

$$\hat{P}_F = \frac{1}{N_R} \sum_{i=1}^{N_R} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}^{(i)}) W(\boldsymbol{\theta}^{(i)}), \tag{9}$$

where $\{\boldsymbol{\theta}^{(i)}; i = 1, \ldots, N_R\}$ are independent samples distributed according to $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. The coefficient of variation (CoV) of the IS estimator depends on the choice of $h_{\mathbf{\Theta}}(\boldsymbol{\theta})$. The theoretically optimal IS density that leads to an estimator with zero variance is given by

$$h^*_{\mathbf{\Theta}}(\boldsymbol{\theta}) = \frac{1}{P_F} P_{F|\mathbf{\Theta}}(\boldsymbol{\theta}) p_{\mathbf{\Theta}}(\boldsymbol{\theta}). \tag{10}$$

In practice, it is not possible to sample from this IS density since P_F is not known. Instead, we construct an IS density for $\mathbf{\Theta}$ that is a close approximation of $h^*_{\mathbf{\Theta}}(\boldsymbol{\theta})$ using the cross entropy (CE) method.

The CE method is an adaptive approach to determine a near-optimal IS density through minimizing the Kullback-Leibler (KL) divergence between the theoretically optimal IS density $h^*_{\mathbf{\Theta}}(\boldsymbol{\theta})$ and a chosen parametric family of probability distributions. Let $h_{\mathbf{\Theta}}(\boldsymbol{\theta}; \boldsymbol{\nu})$ be a family of parametric densities, where $\boldsymbol{\nu} \in \mathcal{V}$ is the parameter vector, such that it contains the nominal density, $p_{\mathbf{\Theta}}(\boldsymbol{\theta})$, of the uncertain parameters . The KL divergence between $h^*_{\mathbf{\Theta}}(\boldsymbol{\theta})$ and $h_{\mathbf{\Theta}}(\boldsymbol{\theta}; \boldsymbol{\nu})$ is defined as (Rubinstein and Kroese 2016)

$$D(h^*_{\mathbf{\Theta}}(\boldsymbol{\theta}), h_{\mathbf{\Theta}}(\boldsymbol{\theta}; \boldsymbol{\nu})) = \mathrm{E}_{h^*_{\mathbf{\Theta}}} \left[\ln \left(\frac{h^*_{\mathbf{\Theta}}(\boldsymbol{\theta})}{h_{\mathbf{\Theta}}(\boldsymbol{\theta}; \boldsymbol{\nu})} \right) \right]. \tag{11}$$

The basic idea of the CE method is to determine the optimal parameter vector $\boldsymbol{\nu}^*$ that minimizes $D(h^*_{\mathbf{\Theta}}(\boldsymbol{\theta}), h_{\mathbf{\Theta}}(\boldsymbol{\theta}; \boldsymbol{\nu}))$. Substitution of the expression of $h^*_{\mathbf{\Theta}}(\boldsymbol{\theta})$ in Eq. (10) into Eq. (11) yields the following CE optimization problem:

$$\boldsymbol{\nu}^* = \underset{\boldsymbol{q} \in \mathcal{V}}{\operatorname{argmax}} \, \mathrm{E}_{p_{\mathbf{\Theta}}} \left[P_{F|\mathbf{\Theta}}(\boldsymbol{\theta}) \ln \left(h_{\mathbf{\Theta}} \left(\boldsymbol{\theta}; \boldsymbol{q} \right) \right) \right]. \tag{12}$$

The above optimization can be solved by approximating the expectation in Eq. (12) using a set of samples distributed according to $p_{\mathbf{\Theta}}(\boldsymbol{\theta})$. However, in practical applications, the optimal density $h^*_{\mathbf{\Theta}}(\boldsymbol{\theta})$ can differ significantly from $p_{\mathbf{\Theta}}(\boldsymbol{\theta})$, in which case a large number of samples is required to obtain a good sample approximation. This difficulty can be circumvented by adopting a multi-level approach (Rubinstein and Kroese 2016). For the case of component reliability problems, i.e., when the structure failure event F is comprised of a single first-passage failure event, we have developed an efficient multi-level strategy to determine the IS density of $\boldsymbol{\Theta}$ by the CE method (Kanjilal et al. 2021). Here we extend this method to deal with series systems.

Multi-level CE method

The multi-level CE method solves the optimization problem in Eq. (12) by defining a sequence of target densities $\{h_{\boldsymbol{\Theta}}^{[k]}(\boldsymbol{\theta}), k = 0, \ldots, L\}$, which starts from the nominal density $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and gradually approaches the optimal IS density $h_{\boldsymbol{\Theta}}^{*}(\boldsymbol{\theta})$. Consider the sequence of PDFs defined according to the expression

$$h_{\boldsymbol{\Theta}}^{[k]}(\boldsymbol{\theta}) = \frac{1}{C_k} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}), \tag{13}$$

where $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_L = 1$ and $C_k = \int_{\boldsymbol{\theta} \in \mathbb{R}^{n_\theta}} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$. Note that $h_{\boldsymbol{\Theta}}^{[0]}(\boldsymbol{\theta}) = p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and $h_{\boldsymbol{\Theta}}^{[L]}(\boldsymbol{\theta}) = h_{\boldsymbol{\Theta}}^{*}(\boldsymbol{\theta})$. The parameters $\{\gamma_k, k = 1, \ldots, L\}$ ensure a smooth transition between $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and $h_{\boldsymbol{\Theta}}^{*}(\boldsymbol{\theta})$. In the multi-level approach, the CE optimization is solved sequentially for each of the intermediate target densities, which leads to a sequence of parameter vectors $\{\boldsymbol{v}^{[k]}, k = 1, \ldots, L\}$. The final parameter vector $\boldsymbol{v}^{[L]}$ should approximate well the solution of Eq. (12).

The parameter vector $\boldsymbol{v}^{[k]}$ in each level is estimated by solving a CE optimization problem that minimizes the KL divergence between $h_{\boldsymbol{\Theta}}^{[k]}(\boldsymbol{\theta})$ and $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \boldsymbol{v})$. The objective function of the resulting optimization, i.e., the expectation $\mathrm{E}_{p_{\boldsymbol{\Theta}}}\left[P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k} \ln\left(h_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}; \boldsymbol{q}\right)\right)\right]$, is approximated with IS using a set of samples $\{\boldsymbol{\theta}^{(i)}, i = 1, \ldots, N\}$ distributed according to $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{v}}^{[k-1]})$, where $\hat{\boldsymbol{v}}^{[k-1]}$ is the estimate of $\boldsymbol{v}^{[k-1]}$ determined in the previous level. The stochastic optimization problem to be solved in each intermediate level is therefore given by

$$\hat{\boldsymbol{v}}^{[k]} = \underset{\boldsymbol{q} \in \mathcal{V}}{\operatorname{argmax}} \frac{1}{N} \sum_{i=1}^{N} \widetilde{W}_k\left(\boldsymbol{\theta}^{(i)}, \hat{\boldsymbol{v}}^{[k-1]}\right) \ln\left(h_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}^{(i)}; \boldsymbol{q}\right)\right), \tag{14}$$

with $\widetilde{W}_k\left(\boldsymbol{\theta}, \hat{\boldsymbol{v}}^{[k-1]}\right) = P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k} \frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{v}}^{[k-1]})}$. A default choice for $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{v}}^{[0]})$ is the nominal density

$p_{\Theta}(\theta)$. The smoothing parameter γ_k is selected adaptively such that the sample CoV $\hat{\delta}_{\widetilde{W}_k}$ of the weights $\left\{\widetilde{W}_k\left(\theta^{(i)}, \hat{v}^{[k-1]}\right), i = 1, \ldots, N\right\}$ is equal to a target value δ_{target}:

$$\gamma_k = \operatorname*{argmin}_{\gamma \in (\gamma_{k-1}, 1)} \left(\hat{\delta}_{\widetilde{W}_k}(\gamma) - \delta_{target}\right)^2. \tag{15}$$

The choice of the value of δ_{target} is discussed in (Papaioannou et al. 2016; Papaioannou et al. 2018). In the present study we set δ_{target} to 1.5. It is noted that Eq. (15) is equivalent to requiring that the number of effective samples available to fit the parametric density at each sampling iteration is equal to a target value for a given N (Latz et al. 2018). The effective sample size (ESS) is expressed in terms of the CoV of the weights as ESS $= N / \left(1 + \hat{\delta}^2_{\widetilde{W}_k}(\gamma)\right)$. The adaptive procedure is stopped when the value of γ_k determined based on Eq. (15) is equal to 1. After convergence at the L-th step, the final parameter vector $\hat{v}^{[L]}$ is determined by solving Eq. (14) with $\gamma_L = 1$. The sampling density $h_{\Theta}(\theta; \hat{v}^{[L]})$ is taken as the IS density of Θ for estimating the probability of failure.

Estimation of the conditional probability of failure during CE optimization

Determination of the IS density $h_{\Theta}(\theta; \hat{v}^{[L]})$ requires evaluation of the conditional failure probability $P_{F|\Theta}(\theta)$ for all samples of Θ generated during CE optimization. In principle, one could estimate $P_{F|\Theta}(\theta)$ by MCS, e.g., by using the IS method in (Au and Beck 2001b). However, to ensure smooth convergence of the CE method, the CoV of the IS estimator of $P_{F|\Theta}(\theta)$ should be small. This requires a sufficient number of samples to be used in the estimator, which, in turn, increases the number of evaluations of the dynamical system. To alleviate the increase in computational effort, we employ an analytical approximation of $P_{F|\Theta}(\theta)$ during CE optimization.

Recall that in series system reliability problems, failure occurs when any one of the responses Z_i out-crosses its prescribed threshold z_i^* within the duration T of the random excitation. The analytical approximation we adopt is based on the Poisson hypothesis for the number of out-crossing (Rice 1944; Melchers and Beck 2018). For the discrete-time setting described earlier, this approximation is given by

$$P_{F|\Theta}(\theta) = 1 - \exp\left(-\sum_{k=2}^{n_T} \alpha(t_k; z^*, \theta)\Delta t\right), \tag{16}$$

where $z^* = \{z_1^*, \ldots, z_m^*\}$ is the vector of response thresholds and $\alpha(t_k; z^*, \boldsymbol{\theta})$ is the out-crossing rate of the vector process $\boldsymbol{Z}(t, \boldsymbol{\theta}, \Xi) = \{Z_1(t, \boldsymbol{\theta}, \Xi), \ldots, Z_m(t, \boldsymbol{\theta}, \Xi)\}$ across the m-dimensional double-sided barrier $\{|Z_i| = z_i^*, i = 1, \ldots, m\}$ at time $t = t_k$. This rate is written as the sum of the individual out-crossing rates of the scalar processes over their respective barrier (Li and Melchers 1993; Song and Der Kiureghian 2006)

$$\alpha(t_k; z^*, \boldsymbol{\theta}) = \sum_{i=1}^{m} \alpha_i(t_k; z^*, \boldsymbol{\theta}), \tag{17}$$

where $\alpha_i(t_k; z^*, \boldsymbol{\theta})$ denotes the out-crossing rate of the process $Z_i(t, \boldsymbol{\theta}, \Xi)$ across the surface $S_i = \{\boldsymbol{Z} : |Z_i| = z_i^*, |Z_j| < z_j^* \forall j \neq i\}$ (representing i-th mode of failure) at time $t = t_k$. For double-sided barrier, $\alpha_i(t_k; z^*, \boldsymbol{\theta})$ is further expressed as $\alpha_i(t_k; z^*, \boldsymbol{\theta}) = \alpha_i^+(t_k; z^*, \boldsymbol{\theta}) + \alpha_i^-(t_k; z^*, \boldsymbol{\theta})$, where $\alpha_i^+(t_k; z^*, \boldsymbol{\theta})$ and $\alpha_i^-(t_k; z^*, \boldsymbol{\theta})$ denote the rates of up- and down- crossings of the process $Z_i(t, \boldsymbol{\theta}, \Xi)$ across the thresholds z_i^* and $-z_i^*$, respectively. For a linear system subjected to a zero mean Gaussian process excitation, the response $Z_i(t, \boldsymbol{\theta}, \Xi)$ is a Gaussian random process (this follows directly from Eq. (2)). Furthermore, due to the zero initial condition, the response process has a zero mean. In this situation, $\alpha_i^+(t_k; z^*, \boldsymbol{\theta}) = \alpha_i^-(t_k; z^*, \boldsymbol{\theta})$ holds, which leads to

$$\alpha_i(t_k; z^*, \boldsymbol{\theta}) = 2\alpha_i^+(t_k; z^*, \boldsymbol{\theta}). \tag{18}$$

The up-crossing rate $\alpha_i^+(t_k; z^*, \boldsymbol{\theta})$ is calculated based on the generalized Rice formula (Belyaev 1968)

$$\alpha_i^+(t_k; z^*, \boldsymbol{\theta}) = \int_{S_i^*} \int_0^\infty \dot{z}_i f_{\dot{Z}_i Z_i \widetilde{\boldsymbol{Z}}}(\dot{z}_i, z_i^*, \tilde{z}; t_k) d\dot{z}_i d\tilde{z}, \tag{19}$$

where $\widetilde{\boldsymbol{Z}}$ is the $(m-1)$-dimensional random process obtained from the vector process $\boldsymbol{Z}$ by removing its i-th component, i.e., $\widetilde{\boldsymbol{Z}} = \{Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_m\}$, $S_i^* = \{\widetilde{\boldsymbol{Z}} : |Z_j| < z_j^*\}$ is the $(m-1)$-dimensional subspace defined on the hyperplane of the i-th face $Z_i = z_i^*$, $\dot{Z}_i$ is the time derivative process of Z_i and $f_{\dot{Z}_i Z_i \widetilde{\boldsymbol{Z}}}(\cdot; t_k)$ is the joint PDF of $\dot{Z}_i$, Z_i and $\widetilde{\boldsymbol{Z}}$ at the same time instant. An analytical approach for evaluating the above integral is derived in (Song and Der Kiureghian 2006).

The approach requires repeated conditioning of the PDF $f_{\dot{Z}_i Z_i \widetilde{\mathbf{Z}}}(\cdot; t_k)$ and is practically feasible when m, i.e., the number of components of the series system, is small. Application of this approach within the framework of the CE method has been explored in (Kanjilal et al. 2020). An alternative approach, applicable to systems with large number of components, is to solve $\alpha_i^+(t_k; \boldsymbol{z}^*, \boldsymbol{\theta})$ by obtaining an upper bound for the integral. This bound reduces the multi-dimensional integral in Eq. (19) into a one-dimensional integral over the real line and is given by (Li and Melchers 1993)

$$\alpha_i^+(t_k; \boldsymbol{z}^*, \boldsymbol{\theta}) \le f_{Z_i}(z_i^*; t_k) \int_{-\infty}^{\infty} \left[\sigma_i \phi\left(-\frac{y}{\sigma_i}\right) + y\Phi\left(\frac{y}{\sigma_i}\right)\right] \frac{1}{\beta_i} \phi\left(\frac{y-\mu_i}{\beta_i}\right) dy, \tag{20}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cumulative distribution function and the PDF of a standard normal random variable, respectively. $f_{Z_i}(z_i^*; t_k)$ denotes the marginal pdf of Z_i at $t = t_k$ evaluated at $Z_i = z_i^*$,

$$f_{Z_i}(z_i^*; t_k) = \frac{1}{\sigma_{Z_i}} \phi\left(\frac{z_i^* - \mu_{Z_i}}{\sigma_{Z_i}}\right), \tag{21}$$

σ_i denotes the standard deviation of $\dot{Z}_i | \{Z_i = z_i^*, \bar{\mathbf{Z}} = \bar{z}\}$ at $t = t_k$,

$$\sigma_i = \sqrt{\mathrm{Var}[\dot{Z}_i | Z_i = z_i^*, \bar{\mathbf{Z}} = \bar{z}]} = \sigma_{\dot{Z}_i}\sqrt{1 - \rho_i^2}, \tag{22}$$

and the parameters μ_i and β_i are given by:

$$\begin{aligned} \mu_i &= \mu_{\dot{Z}_i} + \rho_{Z_i \dot{Z}_i} \frac{\sigma_{\dot{Z}_i}}{\sigma_{Z_i}} (z_i^* - \mu_{Z_i}) \\ \beta_i &= \sigma_{\dot{Z}_i} \sqrt{\rho_i^2 - \rho_{Z_i \dot{Z}_i}^2}. \end{aligned} \tag{23}$$

In Eqs. (21)-(23), μ_{Z_i} and $\mu_{\dot{Z}_i}$ denote the mean of Z_i and $\dot{Z}_i$, σ_{Z_i} and $\sigma_{\dot{Z}_i}$ denote the standard deviation of Z_i and $\dot{Z}_i$, $\rho_{Z_i \dot{Z}_i}$ denotes the correlation coefficient of Z_i and $\dot{Z}_i$ and $\rho_i = \sqrt{\Sigma_{\dot{Z}_i \mathbf{Z}}^{\mathrm{T}} \Sigma_{\mathbf{ZZ}}^{-1} \Sigma_{\dot{Z}_i \mathbf{Z}}} / \sigma_{\dot{Z}_i}$. Here $\Sigma_{\mathbf{ZZ}}$ is the covariance matrix of the vector process $\mathbf{Z}$ and $\Sigma_{\dot{Z}_i \mathbf{Z}}$ is the covariance of $\dot{Z}_i$ and $\mathbf{Z}$. The above statatistics are computed at $t = t_k$ by direct analysis of Eq. (3).

In this work, we derive an analytical solution of the integral in Eq. (20) that facilitates faster computation of the upper bound of $\alpha_i^+(t_k; z^*, \boldsymbol{\theta})$. The details of this derivation are provided in Appendix I. By substituting the solution into Eq. (20) and applying Eqs. (17) and (18) we get the following upper bound of the out-crossing rate of the series system

$$\alpha(t_k; z^*, \boldsymbol{\theta}) \leq 2\sum_{i=1}^{m} f_{Z_i}(z_i^*; t_k)\left\{\sqrt{\sigma_i^2+\beta_i^2}\,\phi\left(\frac{\mu_i}{\sqrt{\sigma_i^2+\beta_i^2}}\right)+\mu_i\Phi\left(\frac{\mu_i}{\sqrt{\sigma_i^2+\beta_i^2}}\right)\right\}. \tag{24}$$

Substitution of Eq. (24) into Eq. (16) leads to an upper bound on the conditional failure probability $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$. We evaluate $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ approximately using this upper bound during CE optimization. The resulting procedure for determining the IS density of $\boldsymbol{\Theta}$ is described in Algorithm 1. The analytical approximation reduces the computational cost at the expense of accuracy. However, numerical studies show that the IS density obtained based on this approach gives fairly accurate estimates of the unconditional failure probability. The IS estimator of $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is applied only during reliability estimation (as described in the next section), after the final IS density of $\boldsymbol{\Theta}$ is obtained.

Finally, we remark that the Poisson approximation for the number of out-crossings used in Eq. (16) may not work well for all linear systems. When the threshold levels are small and/or the response processes have a narrow bandwidth, the assumption of independent out-crossings underlying the Poisson approximation is not justified and could result in erroneous estimates. In such cases, the IS density of the uncertain parameters constructed by the CE method will be sub-optimal, which will increase the sampling CoV of the IS estimator of the series system failure probability. This issue can be addressed by applying Vanmarcke's formula for evaluating the out-crossing rate in Eq. (16). Vanmarcke's formula, proposed in (Vanmarcke 1975) and further developed in (Di Paola 1985; Michaelov et al. 1999; Barbato and Conte 2011), provides an improved estimate of the failure probability by taking into account the dependence between the out-crossing events of the scalar responses $\{Z_i(t, \boldsymbol{\theta}, \boldsymbol{\Xi}), i = 1, \ldots, m\}$ across their respective threshold levels. This leads to a modified out-crossing rate for the vector process $\boldsymbol{Z}(t, \boldsymbol{\theta}, \boldsymbol{\Xi})$, deduced by multiplying $\alpha_i(t; z^*, \boldsymbol{\theta})$ in Eq. (17) with a correction term that is equal to the ratio of the out-crossing rates of the response $Z_i(t, \boldsymbol{\theta}, \boldsymbol{\Xi})$ and its envelope process across the threshold z_i^*. The upper bound on the out-crossing rate $\alpha(t; z^*, \boldsymbol{\theta})$ of the system responses in Eq. (24) needs to be modified accordingly. A detailed description of the correction term is provided in (Song and Der Kiureghian 2006).

Algorithm 1: Determination of IS density of $\boldsymbol{\Theta}$ by the multi-level CE method

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input:
- Sample size N.
- Choice of parametric density $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \boldsymbol{\nu})$.
- Target CoV of the weights at each intermediate level, δ_{target}.

initialization:
- Set $k = 0$.
- Select $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{\nu}}^{[0]})$ as the nominal density $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$.

repeat:
- Set $k = k + 1$.
- Generate independent samples $\left\{\boldsymbol{\theta}^{(i)}, i = 1, \ldots, N\right\}$ from $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{\nu}}^{[k-1]})$.
- Evaluate the out-crossing rates $\left\{\alpha(t_k; z^*, \boldsymbol{\theta}^{(i)}), i = 1, \ldots, N\right\}$ for the random samples at the discrete time instants $\{t_k, k = 1, \ldots, n_T\}$ based on the upper bound in Eq. (24).
- Substitute the upper bounds of the out-crossing rates computed in the previous step into Eq. (16) to compute an approximate estimate (an upper bound) of the conditional failure probabilities $\left\{P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}^{(i)}), i = 1, \ldots, N\right\}$.
- Compute the likelihood ratio $\left\{\frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}^{(i)})}{h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}^{(i)}; \hat{\boldsymbol{\nu}}^{[k-1]})}, i = 1, \ldots, N\right\}$ for the random samples.
- Solve the optimization problem in Eq. (15) to determine γ_k.
  Note that the conditional first-passage probabilities and the likelihood ratios computed in the previous steps are used to evaluate the sample CoV of the weights $\left\{\widetilde{W}_k\left(\boldsymbol{\theta}^{(i)}, \hat{\boldsymbol{\nu}}^{[k-1]}\right), i = 1, \ldots, N\right\}$. Further simulations are not needed in this step.
- Determine $\hat{\boldsymbol{\nu}}^{[k]}$ by solving the optimization problem in Eq. (14).

while $\gamma_k < 1$

output:
- $L = k$ and $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{\nu}}^{[L]})$ = IS density of Θ.

Choice of parametric distribution family

The failure domain in series system reliability problems is comprised of multiple important regions, each representing the domain of the component failure events. Due to this, the optimal IS density $h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is typically multi-modal. Therefore, to adequately represent $h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, the parametric density $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \boldsymbol{\nu})$ should also have a multi-modal behavior. We consider two types of mixture distributions as the parametric family: the Gaussian mixture distribution and the von Mises-Fisher-Nakagami mixture distribution. Recall that $\boldsymbol{\Theta} = \{\Theta_1; \ldots; \Theta_{n_\theta}\}$ is the vector of basic random variables that model the uncertain structural parameters. In reliability analysis, it is common practice to consider that the components of $\boldsymbol{\Theta}$ are independent and standard normally distributed. If the structural parameters are mutually dependent and/or follow a non-Gaussian distribution, they can be generated by an iso-probabilistic transformation of independent standard normal random variables (Hohenbichler and Rackwitz 1981; Der Kiureghian and Liu 1986). Therefore, without loss of generality, we assume that $\boldsymbol{\Theta}$ is an $n_{\boldsymbol{\theta}}$-dimensional standard Gaussian random vector, i.e., $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \prod_{j=1}^{n_{\boldsymbol{\theta}}} p_{\Theta_j}(\theta_j)$, where for every j, $p_{\Theta_j}(\theta_j)$ is a one-dimensional standard Gaussian PDF for Θ_j.

Gaussian mixture distribution

The PDF of a Gaussian mixture (GM) model is defined as the sum of a number of Gaussian PDFs, each of them multiplied by a weighing factor:

$$f_{\mathrm{GM}}(\boldsymbol{\theta}; \boldsymbol{\nu}) = \sum_{s=1}^{n_M} \pi_s f_{\mathrm{G}}(\boldsymbol{\theta}; \boldsymbol{\mu}_s, \boldsymbol{\Sigma}_s), \tag{25}$$

where $f_{\mathrm{G}}(\boldsymbol{\theta}; \boldsymbol{\mu}_s, \boldsymbol{\Sigma}_s)$ is the s-th Gaussian PDF with mean $\boldsymbol{\mu}_s$ and covariance matrix $\boldsymbol{\Sigma}_s$ and $\{\pi_s; s = 1, \ldots, n_M\}$ are normalized weights satisfying the condition $\sum_{s=1}^{n_M} \pi_s = 1$. The parameter vector in this case is given by $\boldsymbol{\nu} = \{\pi_s, \boldsymbol{\mu}_s, \boldsymbol{\Sigma}_s; s = 1, \ldots, n_M\}$, where π_s is scalar-valued, $\boldsymbol{\mu}_s$ is a vector of dimension $n_{\boldsymbol{\theta}}$ and $\boldsymbol{\Sigma}_s$ is an $n_{\boldsymbol{\theta}} \times n_{\boldsymbol{\theta}}$ symmetric matrix. This results in a total of $n_M \frac{n_{\boldsymbol{\theta}}(n_{\boldsymbol{\theta}}+3)}{2} + (n_M - 1)$ unknown parameters in the parametric density. The parameter vector is determined in every level of the CE method by solving the optimization problem in Eq.(14). The optimal solution in each level is obtained by substituting $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \boldsymbol{\nu}) = f_{\mathrm{GM}}(\boldsymbol{\theta}; \boldsymbol{\nu})$ in Eq. (14), and equating the gradient of the objective function with respect to the unknown parameters to zero.

For the special case of $n_M = 1$, an exact analytical solution of the optimization problem can be obtained (Rubinstein and Kroese 2016). For the general case of $n_M > 1$, the optimization is solved iteratively using an appropriate numerical scheme. The recent study in (Geyer et al. 2019) employs the fact that the CE optimization problem can be viewed as a weighted maximum likelihood estimation problem to derive a modified expectation-maximization (EM) algorithm. In the present study, we adopt this approach to solve Eq. (14). The EM procedure and the updating rules of $\hat{\boldsymbol{v}}^{[k]}$ for the GM model are described in (Geyer et al. 2019) and are not further discussed here.

von Mises-Fisher-Nakagami mixture distribution

The CE method with Gaussian densities performs poorly in high-dimensional problems, i.e., in problems where the number $n_{\boldsymbol{\theta}}$ of uncertain structural parameters is large. This is due to two reasons: the first is the degeneracy of the importance weight function $W(\boldsymbol{\theta})$ in high-dimensions (Au and Beck 2003; Katafygiotis and Zuev 2008). The second reason is the number of parameters in the GM model, which increases quadratically with $n_{\boldsymbol{\theta}}$. This results in a rapid increase in the number of samples per level N required to obtain an adequate estimate of the optimal parameter values.

Papaioannou et al. (Papaioannou et al. 2019) introduce the von-Mises-Fisher-Nakagami (vMFN) density as an alternative choice of the parametric family in the CE method. This parametric density is more efficient in high-dimensions. For series system reliability analysis, one should use a von-Mises-Fisher-Nakagami mixture (vMFNM), whose PDF is defined in terms of the polar coordinates of the standard normal random vector $\boldsymbol{\Theta}$:

$$f_{\mathrm{vMFNM}}([r\ \boldsymbol{a}];\boldsymbol{v}) = \sum_{s=1}^{n_M} \pi_s f_{\mathrm{vMFN}}([r\ \boldsymbol{a}];\boldsymbol{\mu}_s,\kappa_s,\psi_s,\Omega_s), \tag{26}$$

where the sample pair $\{r\ \boldsymbol{a}\}$ represents the polar coordinates (radius and direction) of $\boldsymbol{\theta}$ and $f_{\mathrm{vMFN}}([r\ \boldsymbol{a}],\boldsymbol{\mu}_s,\kappa_s,\psi_s,\Omega_s)$ is the s-th vMFN density with parameters $\{\boldsymbol{\mu}_s,\kappa_s,\psi_s,\Omega_s\}$ and normalized weight π_s. The vMFN PDF in Eq. (26) is (Papaioannou et al. 2019)

$$f_{\mathrm{vMFN}}([r\ \boldsymbol{a}];\boldsymbol{\mu}_s,\kappa_s,\psi_s,\Omega_s) = f_{\mathrm{N}}(r;\psi_s,\Omega_s) f_{\mathrm{vMF}}(\boldsymbol{a};\boldsymbol{\mu}_s,\kappa_s), \tag{27}$$

where $f_{\mathrm{vMF}}(\boldsymbol{a};\boldsymbol{\mu}_s,\kappa_s)$ is the PDF of a von Mises-Fisher distribution with mean direction $\boldsymbol{\mu}_s$ ($\|\boldsymbol{\mu}_s\| = 1$) and concentration parameter $\kappa_s \geq 0$ and $f_{\mathrm{N}}(r;\psi_s,\Omega_s)$ is the PDF of a Nakagami distribution with shape parameter $\psi_s \geq 0.5$ and spread parameter $\Omega_s > 0$. The analytical expressions of $f_{\mathrm{vMF}}(\boldsymbol{a}; \boldsymbol{\mu}_s, \kappa_s)$ and $f_{\mathrm{N}}(r; \psi_s, \Omega_s)$ can be found in (Wang and Song 2016; Papaioannou et al. 2019).

When the vMFN distribution is used within the CE method, the unknown parameter vector to be estimated by CE optimization is given by $\boldsymbol{\nu} = \{[\boldsymbol{\mu}_s, \kappa_s, \psi_s, \Omega_s]; s = 1, \ldots, n_M\}$. Here all parameters are scalar-valued, with the exception of $\{\boldsymbol{\mu}_s; s = 1, \ldots, n_M\}$, which are vectors of dimension $n_{\boldsymbol{\theta}}$. Thus, the total number of parameters to be estimated in each sampling iteration is $n_M(n_{\boldsymbol{\theta}} + 3) + (n_M - 1)$, which increases only linearly with $n_{\boldsymbol{\theta}}$. The optimal parameter vector in each level of the CE method is determined by substituting $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \boldsymbol{\nu}) = f_{\mathrm{vMFNM}}([r\ \boldsymbol{a}]; \boldsymbol{\nu})$ in Eq. (14) and equating the derivative of the objective function with respect to the unknown parameters to zero. We apply the EM algorithm to solve the CE optimization problem. The EM procedure and the updating rules of $\hat{\boldsymbol{\nu}}^{[k]}$ for the vMFNM are described in (Papaioannou et al. 2019) and are not further discussed here.

ESTIMATION OF PROBABILITY OF FAILURE BY IMPORTANCE SAMPLING

The IS density of the uncertain structural parameters derived in the previous section is applied to estimate the probability of failure of the series system. To this end, we write the unconditional failure probability of Eq. (6) in the modified form

$$P_F = \int_{\boldsymbol{\theta} \in \mathbb{R}^{n_\theta}} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) \frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{\nu}}^{[L]})} h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{\nu}}^{[L]}) d\boldsymbol{\theta}, \tag{28}$$

where $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{\nu}}^{[L]})$ denotes the IS density of the uncertain parameters $\boldsymbol{\Theta}$ obtained using the CE method and $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is the system probability of failure conditional on $\boldsymbol{\Theta} = \boldsymbol{\theta}$. The conditional failure probability is defined in Eq. (7). To ensure reliable and efficient estimation of P_F based on Eq. (28), we evaluate $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ by IS sampling.

Consider the discrete time representation of the dynamical system introduced earlier. Let the failure event $F_{i,k}$ denote out-crossing of the threshold level z_i^* by the i-th absolute structural response, Z_i, at time instant $t = t_k$. From the definition of the system failure event in Eqs. (4) and (5), it follows that occurrence of any one of the elementary failure events $\{F_{i,k}; i = 1, \ldots, m, k = 1, \ldots, n_T\}$ leads to failure of the structure. Hence, the failure event of the series system is an union of the elementary failure events, i.e., $F = \bigcup_{i=1}^{m} \bigcup_{k=1}^{n_T} F_{i,k}$. In order to evaluate $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ by IS, we introduce an IS density of the random vector $\boldsymbol{\Xi}$ characterizing the input excitation, and modify the integral in Eq. (7) to

$$P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \int_{\boldsymbol{\xi}\in\mathbb{R}^{n_\xi}} \mathrm{I}\left\{(\boldsymbol{\theta},\boldsymbol{\xi}) \in F\right\} \frac{p_{\boldsymbol{\Xi}}(\boldsymbol{\xi})}{h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})} h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) d\boldsymbol{\xi}, \tag{29}$$

where $h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})$ denotes the IS density of $\boldsymbol{\Xi}$ conditional on $\boldsymbol{\Theta} = \boldsymbol{\theta}$. Since the structure is deterministic for a given value of the uncertain parameters, the sampling density $h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})$ can be designed based on available IS methods for dynamic reliability estimation of deterministic structures. For the particular case of deterministic linear structures subjected to Gaussian process excitation, an efficient IS density of $\boldsymbol{\Xi}$ is suggested in (Au and Beck 2001b). We employ this IS density to evaluate $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$. Accordingly, we define $h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})$ as a weighted sum of Gaussian PDFs truncated over the domain of the elementary failure events:

$$h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{k=1}^{n_T} w_{i,k}(\boldsymbol{\theta}) p_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\{(\boldsymbol{\theta},\boldsymbol{\xi}) \in F_{i,k}\}) = \sum_{i=1}^{m} \sum_{k=1}^{n_T} w_{i,k}(\boldsymbol{\theta}) \frac{p_{\boldsymbol{\Xi}}(\boldsymbol{\xi})\mathrm{I}\{(\boldsymbol{\theta},\boldsymbol{\xi}) \in F_{i,k}\}}{\Pr\left[F_{i,k}|\boldsymbol{\Theta} = \boldsymbol{\theta}\right]}, \tag{30}$$

where $w_{i,k}(\boldsymbol{\theta})$ are normalized weights given by

$$w_{i,k}(\boldsymbol{\theta}) = \frac{\Pr\left[F_{i,k}|\boldsymbol{\Theta} = \boldsymbol{\theta}\right]}{\sum_{r=1}^{m} \sum_{s=1}^{n_T} \Pr\left[F_{r,s}|\boldsymbol{\Theta} = \boldsymbol{\theta}\right]}. \tag{31}$$

The probability of occurrence of $F_{i,k}$ conditional on $\boldsymbol{\Theta} = \boldsymbol{\theta}$ is calculated according to the expression $\Pr\left[F_{i,k}|\boldsymbol{\Theta} = \boldsymbol{\theta}\right] = 2\Phi(-h_i^*/\|\boldsymbol{r}_{i,k}(\boldsymbol{\theta})\|)$, where $\boldsymbol{r}_{i,k}(\boldsymbol{\theta})$ is as defined in Eq. (3).

Substituting $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ in Eq. (28) with the integral in Eq. (29), we obtain the following expression for P_F:

$$P_F = \int_{\boldsymbol{\theta}\in\mathbb{R}^{n_\theta}} \int_{\boldsymbol{\xi}\in\mathbb{R}^{n_\xi}} \left\{ \frac{\tilde{P}(\boldsymbol{\theta})}{\sum_{i=1}^{m} \sum_{k=1}^{n_T} \mathrm{I}\{(\boldsymbol{\theta},\boldsymbol{\xi}) \in F_{i,k}\}} W(\boldsymbol{\theta}) \right\} h_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta},\boldsymbol{\xi}) d\boldsymbol{\xi} d\boldsymbol{\theta}, \tag{32}$$

where $h_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta},\boldsymbol{\xi}) = h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{v}}^{[L]})$ is the joint IS density of $\boldsymbol{\Theta}$ and $\boldsymbol{\Xi}$, $W(\boldsymbol{\theta}) = \frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\hat{\boldsymbol{v}}^{[L]})}$ is the importance weight function associated with $\mathbf{\Theta}$ and $\tilde{P}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{k=1}^{n_T} \Pr\left[F_{i,k}|\mathbf{\Theta} = \boldsymbol{\theta}\right]$ is the sum of probabilities of the elementary failure events $\{F_{i,k}; i = 1, \ldots, m, k = 1, \ldots, n_T\}$ conditional on $\mathbf{\Theta} = \boldsymbol{\theta}$. The probability of failure of the series system is therefore estimated by IS as

$$\hat{P}_F = \frac{1}{N_R} \sum_{j=1}^{N_R} \frac{\tilde{P}(\boldsymbol{\theta}^{(j)})}{\sum_{i=1}^{m} \sum_{k=1}^{n_T} \mathrm{I}\{(\boldsymbol{\theta}^{(j)}, \boldsymbol{\xi}^{(j)}) \in F_{i,k}\}} W(\boldsymbol{\theta}^{(j)}), \tag{33}$$

where $\left\{\left(\boldsymbol{\theta}^{(j)}, \boldsymbol{\xi}^{(j)}\right), j = 1, \ldots, N_R\right\}$ are independent samples of the structural parameters and excitation distributed according to $h_{\mathbf{\Theta},\mathbf{\Xi}}(\boldsymbol{\theta}, \boldsymbol{\xi}) = h_{\mathbf{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) h_{\mathbf{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{\nu}}^{[L]})$. In order to generate a sample $\left(\boldsymbol{\theta}^{(j)}, \boldsymbol{\xi}^{(j)}\right)$ from $h_{\mathbf{\Theta},\mathbf{\Xi}}(\boldsymbol{\theta}, \boldsymbol{\xi})$, we first generate $\boldsymbol{\theta}^{(j)}$ from the IS density $h_{\mathbf{\Theta}}(\boldsymbol{\theta}; \hat{\boldsymbol{\nu}}^{[L]})$. The corresponding sample $\boldsymbol{\xi}^{(j)}$ is then generated from the conditional IS density $h_{\mathbf{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}^{(j)})$ according to the algorithm described in (Au and Beck 2001b; Kanjilal et al. 2021).

NUMERICAL INVESTIGATIONS

We investigate the performance of the CE-based IS (CE-IS) method by means of two numerical examples. The first considers a two-story linear shear frame, with two uncertain structural parameters, subjected to a stationary Gaussian white noise. The system failure event is defined in terms of three components. This constitutes a simplified problem in terms of both the number of uncertain structural parameters and component failure modes, and is intended to illustrate different aspects of the proposed method. The second example considers a six-story three-bay moment-resisting frame driven by a filtered non-stationary Gaussian process excitation. This problem demonstrates the performance of the method in a more complicated setting where the system consists of 22 uncertain structural parameters and 24 components. In both examples, $\mathbf{\Theta}$ is a vector of independent standard normal random variables. The uncertain structural parameters are generated from $\mathbf{\Theta}$ by means of iso-probabilistic transformations.

The performance of the CE-IS method is assessed in terms of the sample mean and sample CoV of the estimates of P_F, denoted by $\hat{P}_F$ and $\delta_{\hat{P}_F}$ in this section, and in terms of the number of dynamical system evaluations required by the method. N_{CE} denotes the total number of samples of $\mathbf{\Theta}$ needed to determine the IS density of the uncertain parameters using the CE method. N_R

denotes the number of samples of $(\mathbf{\Theta}, \mathbf{\Xi})$ used to obtain a sample estimate of P_F during reliability estimation, i.e., for evaluating Eq. (33). The dynamical system is required to be evaluated for every sample realization of $\mathbf{\Theta}$ to determine the impulse response functions. During CE optimization, the impulse response functions of the critical responses Z_i and their velocities $\dot{Z}_i$ are post-processed to evaluate the analytical approximation of $P_{F|\mathbf{\Theta}}(\boldsymbol{\theta})$. In the reliability estimation step, the impulse response functions of Z_i are convoluted with a sample realization of the input excitation to obtain a realization of the response time-histories. In the considered examples, the input excitation is represented by a scalar Gaussian process, i.e., $l = 1$. Hence, for every generated sample of the uncertain parameters, the impulse response functions of Z_i and $\dot{Z}_i$ are obtained from a single dynamic analysis. Therefore, N_{CE} and N_R also indicate the number of dynamical system evaluations needed in the CE optimization step and the reliability estimation step, respectively. $N_T = N_{CE} + N_R$ is the total number of system evaluations required to obtain an estimate of P_F. The performance measures are averaged over 50 independent simulation runs. The reference values of the probability of failure are obtained by large-scale direct MCS.

While implementing the CE-IS method, the sample size N_R in the reliability estimation step is selected using two approaches. In the first approach, N_R is taken equal to the number of samples per level for CE optimization, i.e., $N_R = N$. In the second approach, N_R is adapted on the fly to ensure that an estimate of the CoV of the IS estimate of P_F is smaller than a specified target value $\delta^*_{\hat{P}_F}$. The adaptive variant of the IS estimator is implemented according to the procedure described in (Kanjilal et al. 2021).

A two-story linear shear frame

The first example involves a two-story linear shear frame which is excited by a stochastic ground acceleration. The structure, idealized as a mass-spring-dashpot system with 2 degrees of freedom, is depicted in Fig. 1. The system has been previously studied in (Valdebenito et al. 2014). Each floor possesses a mass of $m = 30$Mg. The stiffness parameters $\{k_i; i = 1, 2\}$ are modeled as independent uniform random variables with marginal distribution $k_i \sim \mathrm{U}[12, 28]$MN/m. A classical damping of 4% is assumed for the two modes. The ground acceleration $f(t)$ is modeled as a stationary Gaussian white noise of duration $T = 15$s and spectral intensity $S = 10^{-4}\text{m}^2/\text{s}^3$. The stochastic excitation is discretized at time intervals of $\Delta t = 0.01$s, i.e., $n_T = 1501$. The random vector $\boldsymbol{\Xi}$ characterizing $f(t)$ consists of the sequence of i.i.d. standard normal random variables $\{\Xi_k, k = 1, \ldots, n_T\}$ that generate the white noise at the discrete time instants, i.e., $\left\{f(t_k) = \sqrt{2\pi S/\Delta t}\,\Xi_k, k = 1, \ldots, n_T\right\}$. Three response measures are considered: Z_1= absolute displacement of the first floor, and Z_2 = inter-story drift between first and second floors and Z_3 = absolute displacement of the top floor. The objective is to estimate the probability that any one of these responses exceeds a corresponding threshold z_i^* over the duration of the random excitation.

We consider two choices of the response thresholds: (i) case 1: $(z_1^*, z_2^*, z_3^*) = (0.006, 0.006, 0.006)$m, this is the case studied in (Valdebenito et al. 2014) and (ii) case 2: $(z_1^*, z_2^*, z_3^*) = (0.004, 0.003, 0.006)$m. The reference value of the probability of failure in both cases is estimated by direct MCS with 10^7 samples. The performance of the CE-IS method is investigated for the following parametric families: single Gaussian (S-G) distribution, single vMFN (S-vMFN) distribution, Gaussian mixture (GM) distribution and vMFN mixture (vMFNM) distribution. For the mixture models, $n_M = 3$ densities are considered to account for the three component failure modes of the series system. In the present example, where the number of uncertain structural parameters is $n_{\boldsymbol{\theta}} = 2$, the parameter vector $\boldsymbol{\nu}$ for S-G and S-vMFN distributions consists of 5 unknown parameters, whereas for GM and vMFNM distributions it consists of 17 unknown parameters.

Fig. 2 shows samples from the IS densities of $\boldsymbol{\Theta}$ for case 1 obtained using the different parametric densities. The IS densities are fitted using $N = 250$ samples per level during CE optimization. It is seen that the optimal IS density in this case is uni-modal. This can be attributed to the fact that the contribution to the system failure probability comes primarily from one of the three components, i.e., there is one dominant component failure mode (the top floor displacement) for the considered values of the response thresholds. As a consequence, a uni-model parametric density is able to adequately represent the important region of the failure domain in the $\boldsymbol{\Theta}$-space. The use of mixture distributions does not offer any additional advantage in this case. This is further substantiated by Table 1, where we report the results of reliability analysis and the computational effort for N =

250 and 1000 samples per level. All four parametric densities require two steps on average to converge to the failure domain, as is indicated by the value of N_{CE}. The computational effort required to determine the IS density of $\boldsymbol{\Theta}$ is comparable among the different parametric families. The probability of failure estimates in Table 1 are obtained using a fixed number of samples (equal to N) in the IS estimator, i.e., N_R = number of samples per level during CE optimization. The reference value of the probability of system failure is 1.79×10^{-3} with a CoV of 0.8%. For all four parametric densities, the sample mean of the probability estimates obtained by the CE-IS method compare well with the reference solution. In terms of the sample CoV of the estimates, the performance of the method is similar for all choices of the parametric density.

The IS densities of $\boldsymbol{\Theta}$ in case 2, i.e., for $(z_1^*, z_2^*, z_3^*) = (0.004, 0.003, 0.006)$m, are illustrated in Fig. 3. The failure domain has multiple important regions, as is indicated by the multi-modal nature of the optimal IS density $h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. For the GM and vMFNM distributions, it can be seen that the three mixture components can describe the failure domain sufficiently accurate and that majority of the samples are located near the modes of the optimal IS density, which are the regions that have a higher contribution to the probability of failure. In contrast, the samples from the S-G and S-vMFN distributions are more dispersed with a higher fraction of these located in the less important regions, i.e., regions which have less contribution to the failure probability. As a consequence, the uni-modal parametric densities are less efficient than the mixture distributions for this case. This is further substantiated by the simulation results in Table 2. The results are obtained with $N = 250$ samples per level. The values of N_{CE} indicate that for all choices of the parametric density, the CE method requires two steps on average to converge. We evaluate the IS estimator for P_F with both non-adaptive and adaptive selection of N_R. The two choices of N_R are indicated by N_R-NonAdap and N_R-Adap in Table 2. The results for N_R-Adap correspond to a target CoV of $\delta^*_{\hat{P}_F} = 0.05$. The reference value of the probability of system failure is 4.70×10^{-3} with a CoV of 0.5%. The sample mean of the probability estimates obtained with the two choices of N_R are similar for all parametric densities. Although we observe a small bias in comparison with the reference solution, the estimates given by the CE-IS method are sufficiently accurate for practical use. In terms of the sample CoV of the estimates and the computational effort, the mixture distributions perform better. For N_R-NonAdap, it is seen that the CoV of the estimates obtained using the GM and vMFNM distributions are smaller than the ones obtained using the S-G and S-vMFN distributions. For N_R-Adap, the GM and vMFNM distributions require lesser number of dynamic system evaluations to converge to the target CoV of 5%. The superior performance of the mixture distributions is due to the greater accuracy in describing the multi-modal nature of the failure domain.

We investigate the effect of the sample size per level N on the performance of the method. For this, different values of N in the range 125-1000 are considered. The study is conducted for case 2 using the mixture distributions as the parametric family. The sample means of the probability estimates are similar to those given in Table 2 and hence are not reported separately. The sample CoV of the estimates and the computational effort is depicted in Fig. 4. It is observed that the number of levels required for the CE optimization to converge remains the same (on average equal to two) for all values of N. Hence, the computational effort needed for optimization, N_{CE}, increases monotonically with N. The difference between the vertical coordinates of the dotted line and the solid lines corresponds to N_R, the average number of dynamical system evaluations used in the reliability estimation step. With increase in N, the number of effective samples of $\boldsymbol{\Theta}$ available to fit the parametric densities at each intermediate level increases. This leads to better estimation of the parameters in the IS density of $\boldsymbol{\Theta}$. For N_R-NonAdap, where $N_R = N$, an increase in N also implies an increase in the number of samples of $(\boldsymbol{\Theta}, \boldsymbol{\Xi})$ used to obtain a sample estimate of P_F during reliability estimation. Due to these factors, the sample CoV of the probability estimates for N_R-NonAdap decreases as N increases. For N_R-Adap, it is seen that the sample size for reliability estimation initially decreases as N increases. This is due to the sub-optimality in the IS density of $\boldsymbol{\Theta}$ obtained with a small N, which leads to a greater computational effort during reliability estimation necessary to meet the prescribed $\delta^*_{\hat{P}_F}$. As N increases, one obtains improved estimates of the parameter vector, and the number of samples for reliability estimation starts decreasing. Beyond a certain value of N, $N = 500$ in this example, N_R is nearly constant, which indicates that the IS density of $\boldsymbol{\Theta}$ obtained using 500 samples per level is sufficiently optimal, and a further increase in N does not give any additional advantage during reliability estimation. The sample CoV of the probability estimates for N_R-Adap remains close to the prescribed $\delta^*_{\hat{P}_F}$ for all N. Finally, Fig. 4 shows that the IS estimator with adaptive selection of N_R requires a smaller number of dynamical system evaluations to meet a prescribed CoV. It is seen that the GM with N_R-Adap requires only 1350 system evaluations to achieve a sample CoV less than 5%, whereas with N_R-NonAdap similar accuracy is obtained with approximately 3000 system evaluations. A similar observation is made for the vMFNM distribution. This indicates that if the goal is to achieve a desired value of the sample CoV, the adaptive variant of the IS estimator is more efficient provided that the number of samples per level N is chosen appropriately. Similar results as in Fig. 4 are observed for component reliability analysis of randomly excited uncertain linear structures where the failure event is defined by the first-passage of a single critical response across a prescribed threshold (Kanjilal et al. 2021).

A moment-resisting steel frame

We consider the six-story three-bay moment-resisting steel frame shown in Fig. 5. The structure has been previously analysed in an example given in (Au and Beck 2001b), where deterministic structural parameters are considered. The frame is represented by a two-dimensional linear finite element model. The members connecting the joints of the frame are described by two-noded beam elements with two translational DOF and one rotational DOF per node. The equation of motion of the structure is obtained after applying static condensation wherein only the DOFs representing the horizontal displacement of the columns are retained. The frame members have different cross-sections, which are denoted by $\{C_i; i = 1, \dots, 6\}$ (for columns) and $\{G_i; i = 1, \dots, 3\}$ (for girders) in Fig. 5. For each floor, the same section is used for all the girders. The member sections are taken from Example 2 in (Au and Beck 2001b). The Young's modulus of the members vary with cross-section: $\{E_i; i = 1, \dots, 3\}$ denote the modulus of the girder sections $\{G_i; i = 1, \dots, 3\}$ and $\{E_i; i = 4, \dots, 9\}$ denote the modulus of column sections $\{C_i; i = 1, \dots, 6\}$. $\{E_i; i = 1, \dots, 9\}$ are modeled by independent log-normal random variables with mean 200 GPa and CoV 10%. A lumped mass model is applied, wherein the mass of the frame members and the contribution from the dead loads are lumped at the nodes of the frame. These point masses are considered as uncertain and are modeled by log-normal random variables with mean values given in Table 3 and CoV 10%. Rayleigh damping is assumed so that the first two modes have the same critical damping ratio, which is modeled by a log-normal random variable with mean 0.04 and CoV 10%. Hence, the number of uncertain structural parameters is $n_{\boldsymbol{\theta}} = 22$.

The structure is excited by a stochastic ground acceleration $f(t)$ applied in the horizontal direction. We adopt the characterization of the random excitation given in (Au and Beck 2001b) and model $f(t)$ by a modulated Clough-Penzin filtered white noise:

$$f(t) = \omega_d^2 x_d(t) + 2\eta_d\omega_d\dot{x}_d(t) - \omega_g^2 x_g(t) - 2\eta_g\omega_g\dot{x}_g(t), \tag{34}$$

where $\left\{x_d(t) \;\; \dot{x}_d(t) \;\; x_g(t) \;\; \dot{x}_g(t)\right\}^{\mathrm{T}}$ are the states of the filter defined by the linear system:

$$\begin{aligned}
&\ddot{x}_d(t) + 2\eta_d\omega_d\dot{x}_d(t) + \omega_d^2 x_d(t) = e(t)N(t)\\
&\ddot{x}_g(t) + 2\eta_g\omega_g\dot{x}_g(t) + \omega_g^2 x_g(t) = 2\eta_d\omega_d\dot{x}_d(t) + \omega_d^2 x_d(t)\\
&x_d(0) = 0,\; \dot{x}_d(0) = 0,\; x_g(0) = 0,\; \dot{x}_g(0) = 0.
\end{aligned} \tag{35}$$

In the above equation, $N(t)$ is a Gaussian white noise with zero mean and spectral intensity $S = 1 \times 10^{-3}\mathrm{m}^2/\mathrm{s}^3$. The numerical values of the filter parameters are taken to be $\omega_d = 15.7$ rad/s, $\eta_d = 0.6$, $\omega_g = 17.5$ rad/s and $\eta_g = 0.8$. The envelope function is given by $e(t)$:

$$e(t) = \begin{cases} 0 & \text{for } t \leq 0\text{s} \\ (t/4)^2 & \text{for } 0\text{s} \leq t \leq 4\text{s} \\ 1 & \text{for } 4\text{s} \leq t \leq 14\text{s} \\ \exp(-(t-14)^2/2) & \text{for } t \geq 14\text{s} \end{cases} \tag{36}$$

A duration of $T = 30$s and a sampling time interval of $\Delta t = 0.02$s are used in computing the response of the structure. Therefore, the total number of standard Gaussian random variables in the discrete approximation of $f(t)$ is $n_T = 1501$.

Peak inter-story drift ratio

Here we consider the probability that the peak inter-story drift ratio at any column exceeds a specified threshold level z^* in (%). The responses $\{Z_i; i = 1, \dots, m\}$ thus consist of the inter-story drift ratios of all columns connecting the floors, resulting in $m = 24$ critical responses. As the number of uncertain structural parameters is high, the vMFN density is selected as the parametric family in the CE-IS method. Since all columns in a floor experience nearly the same inter-story drift, a significant overlap of the respective failure domains is expected. To adequately describe the failure domains of all columns in the six stories, a mixture distribution with $n_M = 6$ components is considered. The parameter vector $\boldsymbol{\nu}$ thus consists of 155 unknown parameters.

The simulation results for threshold levels $z^* = 0.5, 0.75$ and 1% are given in Table 4. The IS density of $\boldsymbol{\Theta}$ is determined using $N = 500$ samples per level during CE optimization. The values of N_{CE} reported in the table indicate that the number of levels required for the optimization to converge increases with the threshold level. The probability of failure is estimated using both choices of N_R. The results for N_R-Adap correspond to a target CoV of $\delta^*_{\hat{P}_F} = 0.10$. Failure probability estimates for both N_R-NonAdap and N_R-Adap are comparable and agree well with the reference solution.The sample CoV of the probability estimates for N_R-Adap remain close to the target value $\delta^*_{\hat{P}_F} = 0.10$. For N_R-NonAdap, the probability estimates for higher threshold levels have smaller CoV than for lower thresholds, whereas for N_R-Adap the number of dynamical system evaluations required to achieve the target CoV decreases with increase in the threshold level. For $z^* = 0.5$% the method performs poorly; crude MCS would result in a similar CoV with a lower computational effort of approximately 625 samples. The poor performance of the CE-IS method for $z^* = 0.5$ % can be due to two reasons. First, the out-crossing rate-based analytical approximation is used to evaluate the conditional probability $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ during CE optimization. It is known that the Poisson assumption of the number of out-crossing can perform poorly for low threshold levels. This leads to a sub-optimal IS density of the uncertain structural parameters for $z^* = 0.5$ %. The second reason could be that the applied distribution model might not be able to approximate well the optimal IS density. The latter issue can be addressed by increasing the number of terms in the mixture distribution.

Peak floor acceleration

Here we consider the failure probability that the peak floor acceleration over all stories exceeds a specified threshold level z^* (in g). Since the horizontal displacement of the girders is obtained by linear interpolation of the nodal displacements of the beam elements, this probability is equal to the failure probability that the absolute horizontal acceleration at any one of the nodes of the frame exceeds the threshold level z^*. There are thus $m = 24$ critical responses $\{Z_i; i = 1, \ldots, m\}$ corresponding to the horizontal acceleration at the 24 nodes of the structure. The probability of failure is estimated for threshold levels $z^* = 0.2, 0.3$ and 0.4g. The simulation results obtained using a vMFNM distribution with $n_M = 6$ mixture components is given in Table 5. A sample size of $N = 500$ is used per level during CE optimization. Similar to the case of peak inter-story drifts, the results show that the computational effort required to determine the IS density of the uncertain structural parameters increases with the threshold level. For $z^* = 0.2$ and 0.3g, the probability of failure estimates obtained from the CE-IS method agree well with the reference solution. For $z^* = 0.4$g, a small under-estimation is observed; however, for this threshold the sampling uncertainty of the reference solution is significant. It is seen that the sample CoV of the probability estimates for N_R-NonAdap with $N_R = 500$ is comparable to that for N_R-Adap with $\delta^*_{\hat{P}_F} = 0.10$. However, the number of dynamical system evaluations required to obtain the estimates with N_R-Adap is less. This indicates that for $N = 500$ samples per level the IS estimator with adaptive selection of N_R is more efficient.

CONCLUDING REMARKS

This contribution presents an adaptive IS method to estimate the series system reliability of uncertain linear structures subject to Gaussian loading. The main contribution is the introduction of an efficient IS density of the uncertain structural parameters. We determine this IS density by the CE method through minimizing the KL divergence between the theoretically optimal IS density and a chosen parametric family of probability distributions. Based on an upper bound on the joint out-crossing rate of the output responses, a closed-form analytical approximation of the system failure probability conditional on a fixed value of the structural parameters is derived. The use of the analytical approximation enables smooth convergence of the CE optimization problem. A joint IS density of the uncertain structural parameters and the random excitation is considered to estimate the probability of failure. The numerical results indicate that the proposed approach is efficient and accurate.

We investigate the performance of alternative parametric distribution models, depending on the number of uncertain structural parameters and failure modes. In series systems, where the structural failure event is a union of multiple first-passage failures, the optimal IS density of the uncertain parameters is usually multi-modal in nature. In such cases, a mixture distribution offers more flexibility in approximating the optimal IS density. This is demonstrated in our numerical studies, where we compare the performance of uni-modal and mixture distribution models from the Gaussian and the vMFN density family. The mixture distributions outperform the uni-modal distributions both in terms of the coefficient of variation of the failure probability estimate and the computational effort. In terms of dimensionality of the problems, i.e., the number of uncertain structural parameters $n_{\boldsymbol{\theta}}$ involved, we note that the proposed method remains applicable in high dimensions. However, fitting the IS density of the uncertain parameters by CE minimization becomes computationally expensive due to increase in the number of unknown parameters in the parametric densities. This increase is quadratic in $n_{\boldsymbol{\theta}}$ for the GM distribution and linear in $n_{\boldsymbol{\theta}}$ for the vMFNM distribution. Hence, to adequately fit the parametric IS density in high dimensions, a larger number of samples will be required in each level of the CE method. The required sample size scales approximately the same as the number of unknown parameters. In small dimensions, the GM and vMFNM distributions exhibit similar efficiency. In high dimensions, the vMFNM distribution is more efficient.

As future research, it is interesting to further develop the method for application to non-linear dynamical systems. In the presence of non-linearity, the structural response processes are non-Gaussian. An extension of the method to non-Gaussian response processes poses two key challenges. The first lies in obtaining an analytical approximation of the failure probability conditional on the structural parameters, which is required to determine the IS density of the uncertain parameters by CE minimization. In this regard, application of the Poisson approximation requires knowledge of out-crossing rates of non-Gaussian response processes, which is not straight-forward to obtain. An estimate of the conditional failure probability for non-linear systems can be obtained, for example, by stochastic averaging (dos Santos et al. 2019) or tail-equivalent linearization (Fujimura and Der Kiureghian 2007) techniques. However, these methods are computationally too costly for repeated evaluations in the context of the proposed CE-IS method and would need to be adapted. The second challenge lies in constructing an effective IS density of the random excitation to evaluate the conditional failure probability during reliability estimation. Some approaches can be found in (Schuëller et al. 2004a; Kanjilal and Manohar 2019), but also here additional research is necessary to enable their implementation into the proposed approach.

Data Availability Statement

All data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendix I. Evaluation of the upper bound of $\alpha_i^+(t_k; z^*, \boldsymbol{\theta})$

The integral in the upper bound of the up-crossing rate $\alpha_i^+(t_k; z^*, \boldsymbol{\theta})$ in Eq. (20) is given by

$$J = \int_{-\infty}^{\infty} \left[\sigma_i \phi \left(-\frac{y}{\sigma_i} \right) + y \Phi \left(\frac{y}{\sigma_i} \right) \right] \frac{1}{\beta_i} \phi \left(\frac{y - \mu_i}{\beta_i} \right) dy, \tag{37}$$

where σ_i, μ_i and β_i are as defined in Eqs. (22) and (23). In order to derive an analytical solution of J, we consider the change of variables $u = (y - \mu_i)/\beta_i$. Then Eq. (37) can be re-expressed as

$$
\begin{aligned}
J &= \int_{-\infty}^{\infty} \left[\sigma_i \phi \left(-\frac{\beta_i u + \mu_i}{\sigma_i} \right) + (\beta_i u + \mu_i) \Phi \left(\frac{\beta_i u + \mu_i}{\sigma_i} \right) \right] \phi(u) du \\
&= \sigma_i \int_{-\infty}^{\infty} \phi \left(-\frac{\beta_i u + \mu_i}{\sigma_i} \right) \phi(u) du + \beta_i \int_{-\infty}^{\infty} \Phi \left(\frac{\beta_i u + \mu_i}{\sigma_i} \right) u \phi(u) du \\
&\qquad\qquad + \mu_i \int_{-\infty}^{\infty} \Phi \left(\frac{\beta_i u + \mu_i}{\sigma_i} \right) \phi(u) du \\
&= \sigma_i J_1 + \beta_i J_2 + \mu_i J_3
\end{aligned}
\tag{38}
$$

There exist well-known expressions for evaluating integrals of functions of normal densities (Owen 1980). Using these results we get the following analytical expressions for J_1, J_2 and J_3:

$$
\begin{aligned}
J_1 &= \int_{-\infty}^{\infty} \phi \left(-\frac{\beta_i u + \mu_i}{\sigma_i} \right) \phi(u) du = \frac{\sigma_i}{\sqrt{\beta_i^2 + \sigma_i^2}} \phi \left(\frac{\mu_i}{\sqrt{\beta_i^2 + \sigma_i^2}} \right) \\
J_2 &= \int_{-\infty}^{\infty} \Phi \left(\frac{\beta_i u + \mu_i}{\sigma_i} \right) u \phi(u) du = \frac{\beta_i}{\sqrt{\beta_i^2 + \sigma_i^2}} \phi \left(\frac{\mu_i}{\sqrt{\beta_i^2 + \sigma_i^2}} \right) \\
J_3 &= \int_{-\infty}^{\infty} \Phi \left(\frac{\beta_i u + \mu_i}{\sigma_i} \right) \phi(u) du = \Phi \left(\frac{\mu_i}{\sqrt{\beta_i^2 + \sigma_i^2}} \right)
\end{aligned}
\tag{39}
$$

Substitution of Eq. (39) into Eq. (38) leads to the result

$$
J = \sqrt{\beta_i^2 + \sigma_i^2} \phi \left(\frac{\mu_i}{\sqrt{\beta_i^2 + \sigma_i^2}} \right) + \mu_i \Phi \left(\frac{\mu_i}{\sqrt{\beta_i^2 + \sigma_i^2}} \right) \tag{40}
$$

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List of Tables

TABLE 1. Failure probability estimates of two-story linear shear frame - case 1. Results obtained by the CE-IS method using the S-G, GM, S-vMFN and vMFNM distributions. The reference value of probability of failure estimated from 10^7 direct Monte Carlo samples is 1.79×10^{-3} with a CoV of 0.8%

	$N = 250$					$N = 1000$				
	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_{CE}	N_R	N_T	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_{CE}	N_R	N_T
S-G	1.69×10^{-3}	0.071	510	250	760	1.71×10^{-3}	0.032	2000	1000	3000
GM	1.71×10^{-3}	0.068	535	250	785	1.71×10^{-3}	0.036	2020	1000	3020
S-vMFN	1.69×10^{-3}	0.069	510	250	760	1.70×10^{-3}	0.034	2040	1000	3040
vMFNM	1.70×10^{-3}	0.068	520	250	770	1.72×10^{-3}	0.032	2020	1000	3020

TABLE 2. Failure probability estimates of two-story linear shear frame - case 2. Results obtained by the CE-IS method using the S-G, GM, S-vMFN and vMFNM distributions with $N = 250$ samples per level. The results for N_R-Adap correspond to a target CoV of $\delta^*_{\hat{P}_F} = 0.05$. The reference value of probability of failure estimated from 10^7 direct Monte Carlo samples is 4.70×10^{-3} with a CoV of 0.5%

		N_R-NonAdap				N_R-Adap			
	N_{CE}	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_R	N_T	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_R	N_T
S-G	515	4.50×10^{-3}	0.174	250	765	4.38×10^{-3}	0.050	1400	1915
GM	515	4.40×10^{-3}	0.095	250	765	4.41×10^{-3}	0.047	838	1353
S-vMFN	520	4.25×10^{-3}	0.120	250	770	4.43×10^{-3}	0.048	954	1474
vMFNM	530	4.47×10^{-3}	0.106	250	780	4.38×10^{-3}	0.045	842	1372

TABLE 3. Probabilistic description of the point masses in moment-resisting steel frame. The point masses follow log-normal distribution with mean values reported in the table and CoV of 10%.

Floor	Mean value of point mass ($\times 10^3$ kg)	
	Exterior column	Interior column
2	60.4	81.0
3	53.3	78.1
4	51.9	76.0
5	51.7	75.8
6	50.1	73.5
Roof	44.6	63.1

TABLE 4. Failure probability estimates for peak inter-story drift ratio of moment-resisting steel frame. Results from CE-IS method with N_R-Adap correspond to a target CoV of $\delta^*_{\hat{P}_F} = 0.10$. $N = 500$ samples used per level during CE optimization. Reference solution estimated by 2×10^6 direct Monte Carlo samples.

z^* (%)	CE-IS									direct MC	
		N_R-NonAdap				N_R-Adap					
	N_{CE}	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_R	N_T	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_R	N_T	P_F	δ_{P_F}
0.5	500	1.33×10^{-1}	0.173	500	1000	1.28×10^{-1}	0.112	1203	1703	1.31×10^{-1}	0.002
0.75	1000	4.36×10^{-3}	0.137	500	1500	4.21×10^{-3}	0.103	858	1858	4.31×10^{-3}	0.011
1	1620	7.61×10^{-5}	0.142	500	2120	7.32×10^{-5}	0.090	696	2316	7.70×10^{-5}	0.081

TABLE 5. Failure probability estimates for peak floor acceleration of moment-resisting steel frame. Results from CE-IS method with N_R-Adap correspond to a target CoV of $\delta^*_{\hat{P}_F} = 0.10$. $N = 500$ samples used per level during CE optimization. Reference solution estimated by 2×10^6 direct Monte Carlo samples.

z^* (g)	CE-IS									direct MC	
		N_R-NonAdap				N_R-Adap					
	N_{CE}	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_R	N_T	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_R	N_T	P_F	δ_{P_F}
0.2	500	1.09×10^{-1}	0.091	500	1000	1.06×10^{-1}	0.093	392	892	1.09×10^{-1}	0.002
0.3	1480	2.21×10^{-3}	0.112	500	1980	2.14×10^{-3}	0.080	302	1782	2.21×10^{-3}	0.015
0.4	2120	2.42×10^{-5}	0.098	500	2620	2.40×10^{-5}	0.102	476	2596	1.55×10^{-5}	0.180

List of Figures

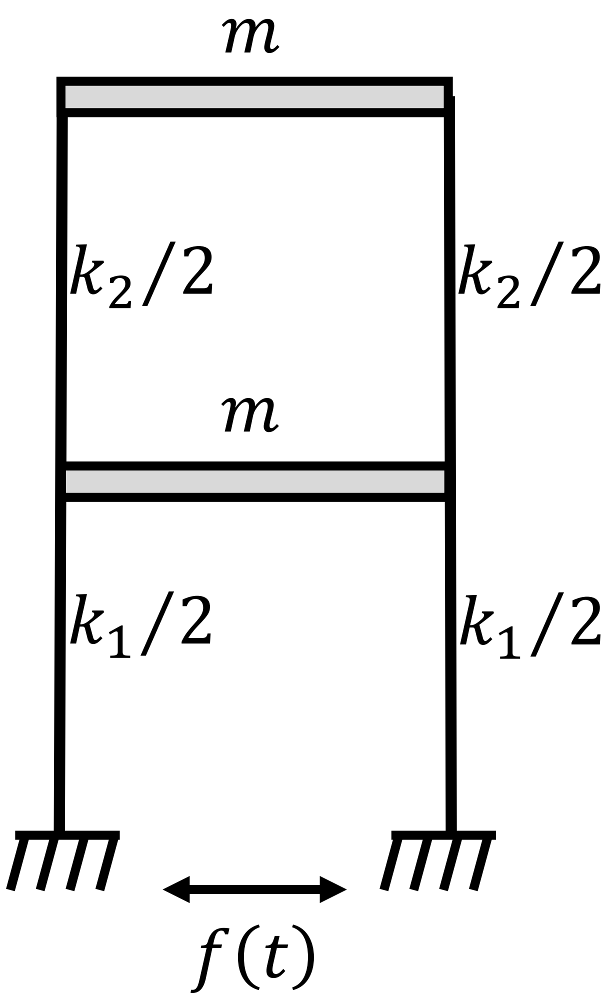

Fig. 1. Two-story shear frame excited by stochastic ground acceleration $f(t)$

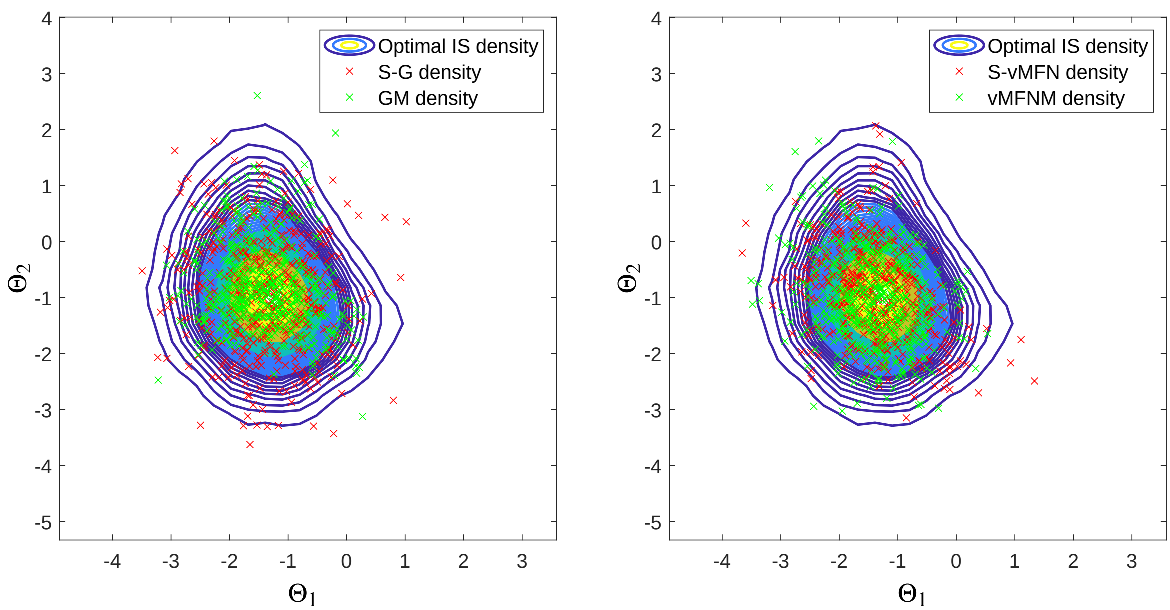

Fig. 2. Comparison of the IS density of the uncertain structural parameters in the standard normal space ($\mathbf{\Theta}$-space) for two-story linear shear frame - case 1. The solid lines represent the contours of the optimal IS density $h^*_{\mathbf{\Theta}}(\boldsymbol{\theta})$, which is estimated by direct MCS with 10^7 samples. The scattered points are samples of $\mathbf{\Theta}$ drawn from the IS density obtained by the CE-IS method.

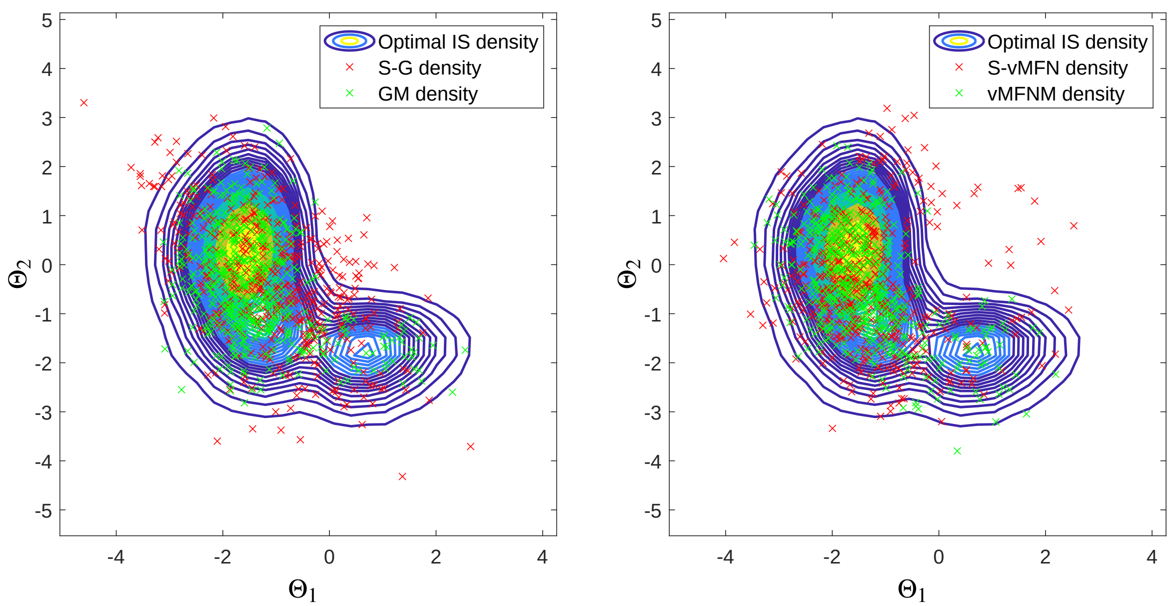

Fig. 3. Comparison of the IS density of the uncertain structural parameters in the standard normal space ($\mathbf{\Theta}$-space) for two-story linear shear frame - case 2. The solid lines represent the contours of the optimal IS density $h^*_{\mathbf{\Theta}}(\boldsymbol{\theta})$, which is estimated by direct MCS with 10^7 samples. The scattered points are samples of $\mathbf{\Theta}$ drawn from the IS density obtained by the CE-IS method.

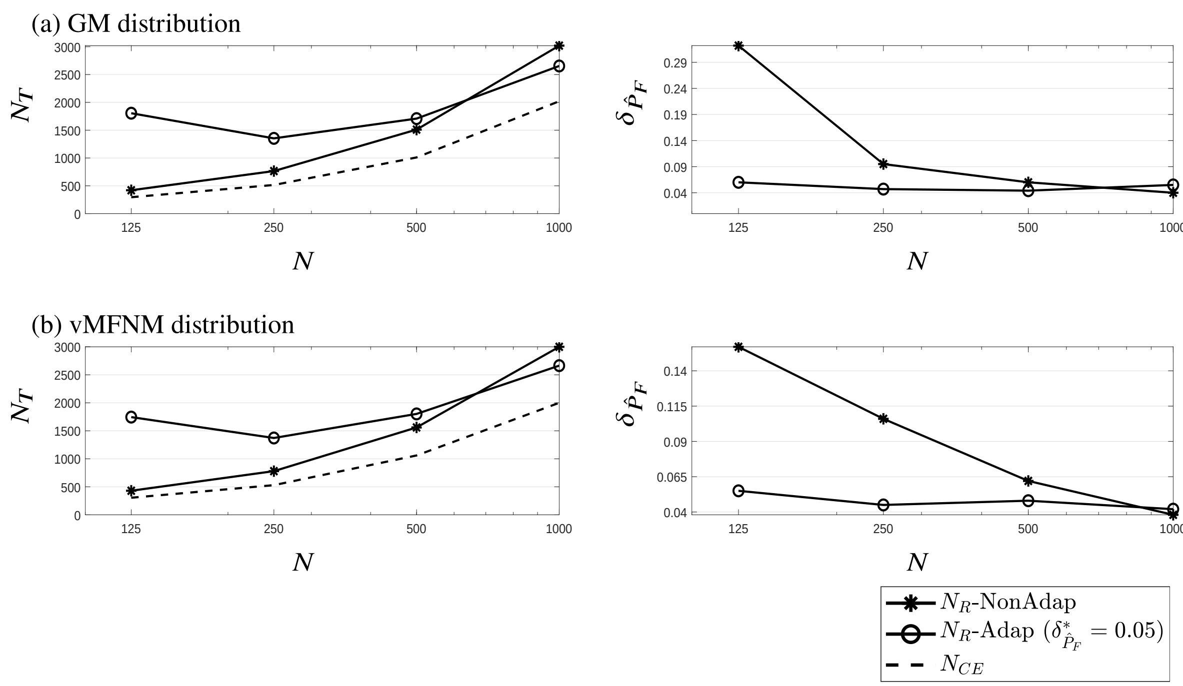

Fig. 4. Total computational effort N_T and the sample CoV $\delta_{\hat{P}_F}$ as a function of N for two-story linear shear frame - case 2. The rows correspond to different parametric densities (a) GM distribution; (b) vMFNM distribution. Note that the dashed line does not reflect N_T, but the computational effort needed only for CE optimization.

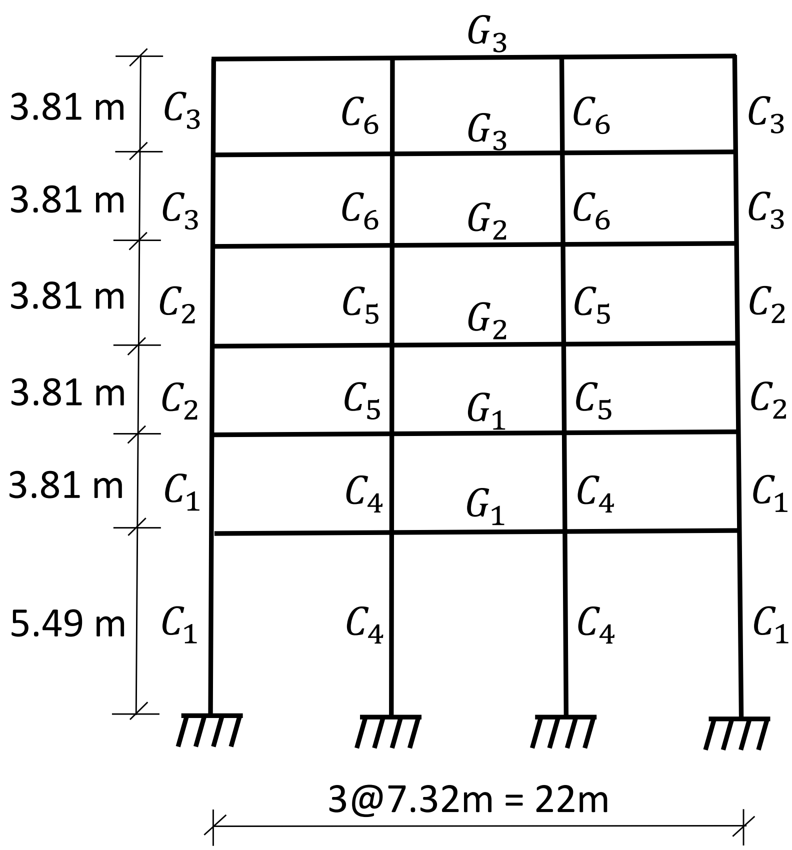

Fig. 5. Moment-resisting steel frame