A generalized Daniels system for reliability assessment of redundant structural systems

Redundancy effects in partial safety factor design: A link to a generalized Daniels system

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Abstract

The partial safety factor concept is applied on element level. Its safety components are calibrated such that a target reliability is – on average – achieved for each element failure mechanism (e.g. bending failure of a beam or stability failure of a column). System effects are not taken into account. Resulting system reliabilities can be below as well as above the target reliability, depending on the redundancy of the system. We introduce an additional partial safety factor which increases the resistances of systems with high amounts of redundancy and decreases the resistance of systems with low amounts of redundancy. The values of the additional partial safety factor are derived via a link to a generalized version of the Daniels system. The generalization includes probabilistic load modeling, material models, correlation among members and non-equal load-sharing among members. We propose efficient reliability evaluation methods and conduct numerical investigations for each generalization.

1. Introduction

The current state of the art designing approach of structures is the partial safety factor (PSF) concept [1–4]. The PSF concept is a semi-probabilistic design approach which is applied at element level. Various element failure mechanism need to be considered individually (e.g. bending failure of a beam or stability failure of a column). System effects are not taken into account. The system reliability achieved by the partial safety factor concept can be below as well as above the reliabilities at element level: In some cases failure at element level may not lead to system failure (e.g. if just one out of many cables of a suspension bridge fails). In other cases element failure directly leads to system failure (e.g. for statically determined structures).

Back when the PSF concept was introduced, the developing community was already aware of this issue; however, "since the knowledge of system reliability is incomplete and not sufficiently documented for practice, a quantitative assessment of various structural systems is not intended" [4]. Today, system reliability is well studied. Various methods exist to evaluate system reliabilities or to take system effects into account within the structural design [5–8]. But none of these methods is applicable within the scope of the partial safety concept.

The objective of this paper is to take system effects into account without leaving the semi-probabilistic framework of the partial safety concept. To do this, we introduce an additional partial safety factor (PSF) which value depends on various system properties. The PSF is applied to each element. It decreases the element resistances of systems with high amount of redundancy and increasing the element resistances of systems with low amount of redundancy. This homogenizes the reliability level with respect of system effects.

To derive the values of the additional PSF we make use of the Daniels system (see Figure 1). The Daniels system is a structural system of n parallel members, which share the same load such that the strain on all members is equal. However, the original version of the Daniels system is not universal enough to represent any structural system. We, therefore, extend the original Daniels system.

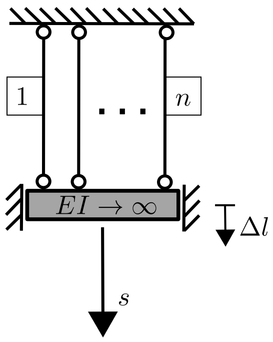

Figure 1: Daniels system.

The idea how to use the Daniels system to derive the additional PSF is, in a nutshell, the following: First, we establish a link between general structural systems and the extended version of the Daniels system, such that we are able to represent the reliability of any structural system in terms of the reliability of a Daniels system. Secondly, we perform a recalibration of the PSF concept with respect to the design of different variations of the Daniels system. The recalibration is conducted by introducing the additional PSF γ_{Sys} for the different variations of the Daniels system. This perfectly homogenizes the reliability level of the Daniels system with respect to system effects. Thirdly, the partial safety factors can be translated back and applied to general structural systems. Due to the necessary simplifications, the derived PSF does not perfectly homogenize the reliability level of general structural systems, however, it is a step in the right direction.

2. The Daniels system

The original formulation of the Daniels system (see Figure 1) has the following properties: Deterministic and equal cross sections A_i and Young's modulus E_i and deterministic load. The horizontal bar distributes the load equally among all members that have not failed. The ultimate strengths $\sigma_{max,i}$ of all members are independent and identically distributed (i.i.d.). Members exhibit linear-elastic brittle material behavior according to Figure 2.

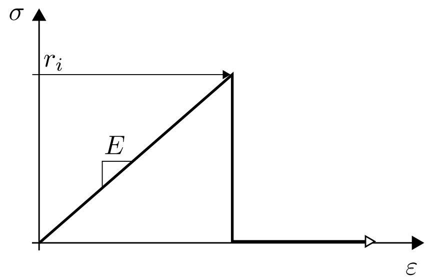

Figure 2: Brittle material behavior of one member of the Daniels system: Linear-elastic material behavior until element strength $\sigma_{max,i} = r_i$ is reached. Once r_i is exceeded, the resistance drops to zero.

For simplification purposes and without loss of generality, we set all cross sections to one: $A_i := 1$ $(i = 1, \ldots, n)$. Consequently, the maximal resistance R_i of the ith bar is equal to $\sigma_{max,i}$.

The resistance of the Daniels system is determined from the following train of thought: The system resistance until the first member fails is n times the resistance of the weakest member. The system resistance until the second member fails is $n-1$ times the resistance of the second weakest member. And so on. It follows that the system resistance $R_{Sys,n}$ of a Daniels system with n members is

$$R_{Sys,n} = \max_{i=1}^{n} \left\{ (n-i+1) \cdot R_{(i)} \right\} \tag{1}$$

where $R_{(i)}$ is the resistance of the ith strongest member, i.e., it is the ith order statistic of R_i. From this the probability of system failure $\Pr(F_{Sys,n})$ can be determined by solving the following integral:

$$\Pr(F_{Sys,n}) = n! \cdot \int_0^{\frac{s}{n}} \int_{r_n}^{\frac{s}{n-1}} \cdots \int_{r_2}^{s} f_R(r_n) \cdot f_R(r_{n-1}) \ldots f_R(r_1) \, \mathrm{d}r_1 \ldots \mathrm{d}r_{n-1} \, \mathrm{d}r_n \tag{2}$$

where f_R is the probability density function (PDF) of the member resistance R.

Daniels found multiple analytical expressions to calculate this integral [9]. The most well-known one is derived via a Taylor series expansion of the multidimensional integral in terms of the lower bound of the first integral and via a recursive relation of the n-folded integral and its derivative. From this, the following recursive formula of the CDF of the system resistance can be derived [10, 11]:

$$F_{R_{Sys,n}}(s) = (-1)^{n+1} \cdot F_R^n\left(\frac{s}{n}\right) - \sum_{i=1}^{n-1} \left[(-1)^i \cdot \binom{n}{i} \cdot F_R^i\left(\frac{s}{n}\right) \cdot F_{R_{Sys,n-i}}(s) \right] \tag{3}$$

where F_R is the CDF of the resistance of a single member.

Remark: Besides the exact formula to calculate the probability of failure, Daniels also found that $R_{Sys,n}$ is asymptotically normally distributed [9]. However, convergence is rather slow.

We extent the Daniels system with respect to four aspects: Probabilistic load modeling, material modeling of the members, correlation among members and non-equal load-sharing among members. For all extensions, we develop efficient algorithms for reliability evaluations, which are fast and accurate. The extensions can be combined. We do not include detailed derivations of each extended version Daniels systems in the main paper, since these are not crucial to understand the main message of the paper. Detailed derivation are given in Appendix A. Moreover, Appendix Aand numerical studies includes numerical studies for each extension. From these the respective effects to the system reliability can be understood and transferred to general structural systems.

3. Recalibration of the PSF concept with respect to system effects

To perform the recalibration we first establish a the linkt from general structural systems to the Daniels system. The following list of entities describes this link. The list is complete in the sence that no further entities need to be known to form extended Daniels systems which –from the point of view reliability of reliability– are equivalent to any general structural system. We do not give a formal proof that this list is complete; however, the equivalents of the list and the extended Daniels system can be understood from the example in Section 4.

1. Number of system failure mechanisms: Equal to the number of extended Daniels systems needed. These Daniels systems are connected as a series system. Failure of one Daniels system is equal to the occurrence of a failure mechanism, hence, system failure.

2. Number of element failures that lead to a system failure mechanism: Number of members in the corresponding Daniels system.

3. Resistance of each element of the structure: Directly corresponds to the material type and the maximal stiffness of the elements of the Daniels system.

4. Correlation among the elements of the structure: Determines the correlation among members of the Daniels system.

5. Loads of the structure: Directly correspond to the load of each Daniels system.

6. Geometry of the structure: From the geometry, the load effects can be calculated, hence, the relation between loads and resistances of the Daniels system can be established. This determines the load-sharing properties of the Daniels system.

The possible combinations of the six entities are endless. In order to derive the additional PSF, some necessary simplifications and assumptions have to be made. The derived PSF is only valid on that basis. The simplifications and assumptions per item of the above list are the following:

1. We assume that one system failure mechanism is dominant and all other system failure mechanisms can be neglected: No series system of Daniels systems has to be evaluated, but only a singular Daniels system.

2. We only account for global failure mechanisms. Under this assumption, the number of element failures that lead to a system failure mechanism is equal to the static over-determination of the system plus one. We investigate Daniels systems with $n = 1,\ldots,10$ members; hence, statically determined and $1,\ldots,9$ times statically over-determined systems.

3. The material behavior of each element of the structure is assumed to be of the same type. No Daniels systems with mixed materials are considered. Further, different dimensions of the elements of the structure are neglected; hence, only Daniels systems with equal stiffnesses per member are investigated. We investigate ideal plastic material ($f_{res} = 1$), ideal brittle material ($f_{res} = 0$), and semi plastic material behavior (modeled following material model 2 with $f_{res} \in \{0.25, 0.5, 0.75\}$).

4. We consider only positive equicorrelation among elements with a correlation coefficient of $\rho_{R_iR_j} \in \{0, 0.3, 0.6, 0.9\}$.

5. We account for the variability of the load side by utilizing a portfolio of load combinations S inspired by the ongoing revision of the Eurocode [12]:

   $$S = (1 - a_Q) \cdot G + a_Q \cdot \Theta_{Q_i} Q_i \tag{4}$$

   where G is the permanent load

   $$G \sim \mathcal{N} \qquad \mathrm{E}[G] = 1.00 \qquad \text{c. o. v.}[G] = 0.10 \tag{5}$$

   Q_i is one of the following variable loads

   $$\text{Wind load:} \quad Q_1 \sim \mathcal{G} \quad \mathrm{E}[Q_1] = 1.00 \quad \text{c. o. v.}[Q_1] = 0.30 \tag{6}$$
   $$\text{Snow load:} \quad Q_2 \sim \mathcal{G} \quad \mathrm{E}[Q_2] = 1.00 \quad \text{c. o. v.}[Q_1] = 0.50 \tag{7}$$
   $$\text{Imposed load:} \quad Q_3 \sim \mathcal{G} \quad \mathrm{E}[Q_3] = 1.00 \quad \text{c. o. v.}[Q_1] = 0.80 \tag{8}$$

   Θ_{Q_i} is the model uncertainty of the respective variable load

   $$\text{Wind load:} \quad \Theta_{Q_1} \sim \mathcal{LN} \quad \mathrm{E}[Q_1] = 0.97 \quad \text{c. o. v.}[Q_1] = 0.26 \tag{9}$$
   $$\text{Snow load:} \quad \Theta_{Q_2} \sim \mathcal{LN} \quad \mathrm{E}[Q_2] = 0.81 \quad \text{c. o. v.}[Q_1] = 0.26 \tag{10}$$
   $$\text{Imposed load:} \quad \Theta_{Q_3} \sim \mathcal{LN} \quad \mathrm{E}[Q_3] = 1.00 \quad \text{c. o. v.}[Q_1] = 0.10 \tag{11}$$

   and a_Q is a constant reflecting the proportion of the dead weight and the variable load. We consider nine different values of a_Q equally distributed within the interval $[0.1, 0.9]$.

6. We neglect the geometry of the structure and assume that the loads evenly distribute among the elements of the structure and also evenly redistribute in case of element failure. In the case of plastic material behavior, this assumption is always satisfied (since we already assumed equal element strength and dimensions); however, in the case of brittle or semi-plastic material behavior, this assumption is critical. In these cases, the derived additional PSF should be applied with caution.

Given a specific Daniels system, we calculate the average reliability index with respect to the portfolio of load cases as:

$$\overline{\beta} = \frac{\sum_{i=1}^{3}\sum_{j=1}^{9}\beta_{Q_i,a_{Q,j}}}{27} \tag{12}$$

where $\beta_{Q_i,a_{Q,j}}$ is the reliability index of the Daniels system loaded by the load case resulting from Q_i and $a_{Q,j}$. Figure 3 shows the resulting average reliability indices.

To homogenize the reliability of Daniels systems we introduce an additional partial safety factor γ_{Sys}. γ_{Sys} is applied to the characteristic value of the resistance in addition to γ_M; hence, the design resistance is calculated as $r_d = \frac{r_k}{\gamma_M \cdot \gamma_{Sys}}$. The value of γ_{Sys} depends on the number of members of the Daniels system n, the plastic residual of the material r, the coefficient of variation of the member resistances c. o. v.$[R]$ and the correlation among members ρ. To determine the values of γ_{Sys} the following equation was solved

$$\overline{\beta}(\gamma_{Sys}; n, r, \text{c. o. v.}[R], \rho) \overset{!}{=} \beta_T = 4.3 \tag{13}$$

Figuratively speaking, the application of γ_{Sys} makes all lines in Figure 3 constant at the target reliability of 4.3. Note that only the average reliability $\overline{\beta}$ is constant, but the individual cases of the considered portfolio still scatter; however, this scattering is not related to system effects anymore.

Table 1 shows the values of γ_{Sys} exemplarily for the case of c. o. v.$[R] = 0.1$ and $\rho = 0.3$. Due to limited space, we do not list all values of γ_{Sys}. The vast majority of values we derived is close to 1. The minimum value we derived is $\gamma_{Sys} = 0.75$ (ideal plastic system with c. o. v.$[R] = 0.2$ and $\rho = 0.0$). The maximum value we derived is $\gamma_{Sys} = 1.10$ (c. o. v.$[R] = 0.2$ and $\rho = 0.6$).

Table 1: γ_{Sys} for c. o. v.$[R] = 0.1$ and $\rho = 0.3$.

n	1	2	3	4	5	6	7	8	9	10
Ideal plastic ($f_{res} = 1.00$)	1.00	0.97	0.96	0.95	0.95	0.95	0.94	0.94	0.94	0.94
Semi-plastic ($f_{res} = 0.75$)	1.00	1.01	1.03	1.03	1.03	1.03	1.03	1.03	1.04	1.04
Semi-plastic ($f_{res} = 0.50$)	1.00	1.02	1.04	1.04	1.05	1.05	1.05	1.06	1.06	1.07
Semi-plastic ($f_{res} = 0.25$)	1.00	1.02	1.04	1.05	1.06	1.06	1.06	1.07	1.07	1.08
Ideal brittle ($f_{res} = 0.00$)	1.00	1.03	1.04	1.05	1.06	1.07	1.07	1.08	1.08	1.08

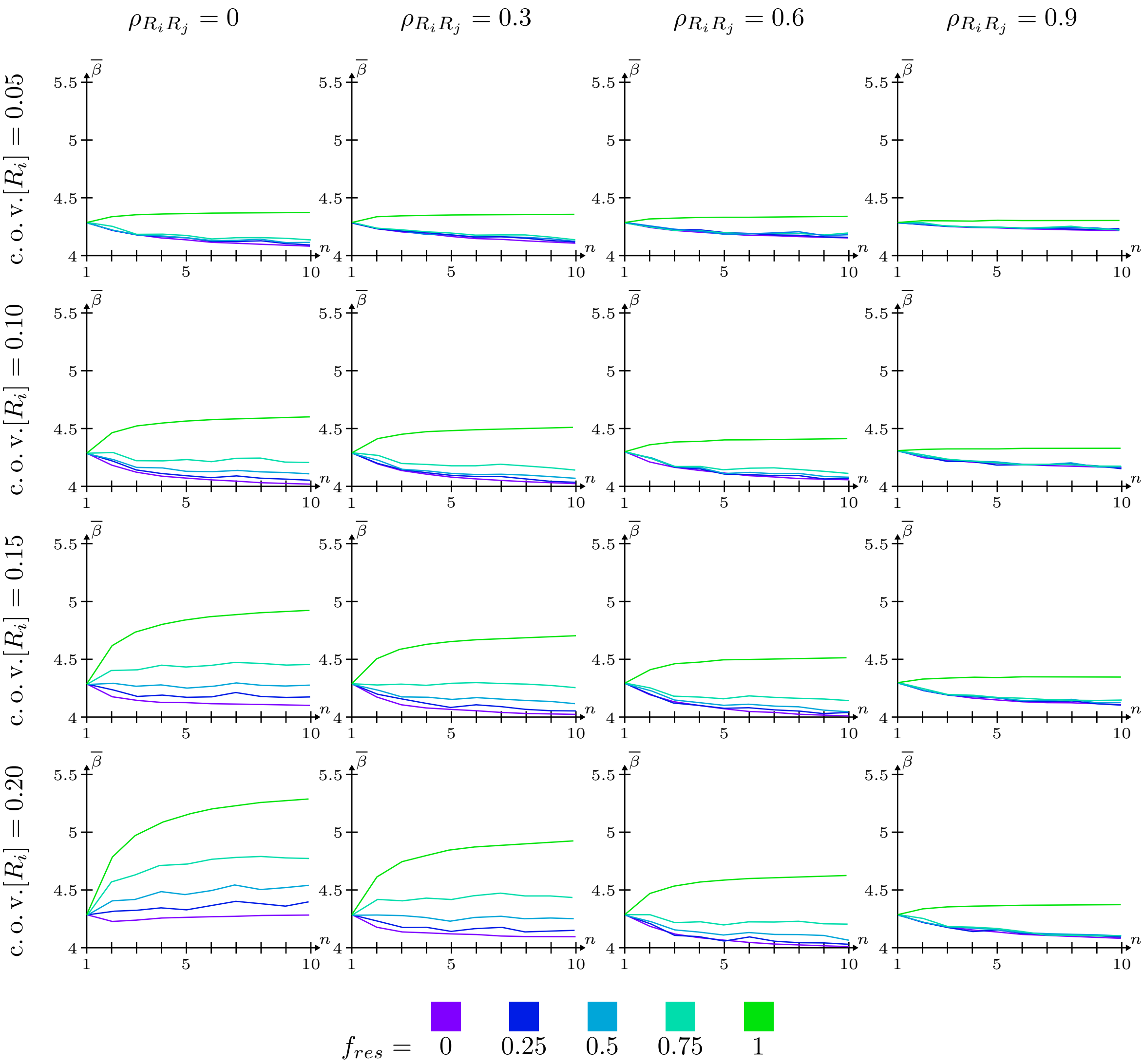

Figure 3: Average reliability indices of Daniels systems with $n = 1, \ldots, 10$ members, $f_{res} \in \{0, 0.25, 0.5, 0.75, 1\}$, c. o. v.$[R_i] \in \{0.05, 0.1, 0.15, 0.2\}$ and $\rho_{R_iR_j} \in \{0, 0.3, 0.6, 0.9\}$.

4. Example structure

In the following, we investigate an –in literature well-known– example structural system from Madsen et al. [13]. We design the structural system following the standard PSF design, perform a reliability analysis, establish a link to the Daniels system, and redesign the structure with the additional PSF γ_{Sys}.

Figure 4 shows the discussed structural system. It is a frame with a horizontal load H and a vertical loading load V. Potential bending failure at 5 different locations is considered. Other element failure mechanisms are not considered.

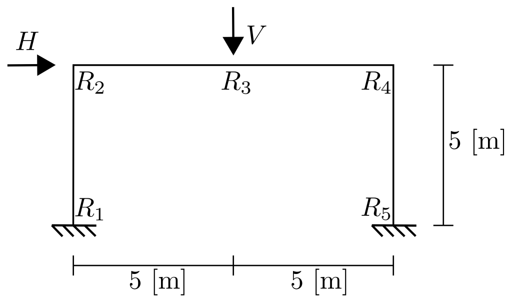

Figure 4: Example frame structure.

We choose V to be a log-normally distributed permanent load ($\mathrm{E}[V] = 1$ and $\mathrm{c.o.v.}[V] = 0.2$) and H to be a Gumbel distributed variable load ($\mathrm{E}[H] = 1$ and $\mathrm{c.o.v.}[H] = 0.3$).

4.1. Determination of the resistances according to the partial safety concept

We choose the characteristic loads as

$$h_k = F_H^{-1}(0.98) = 1.78 \tag{14}$$
$$v_k = \mathrm{E}[V] = 1 \tag{15}$$

where F_H^{-1} is the inverse CDF of H. The corresponding partial safety factors are chosen as

$$\gamma_H = 1.5 \tag{16}$$
$$\gamma_V = 1.35 \tag{17}$$

The resulting design loads are

$$h_d = h_k \cdot \gamma_H = 2.67 \tag{18}$$
$$v_d = v_k \cdot \gamma_V = 1.35 \tag{19}$$

Two load cases are considered: $V \oplus H$ acting simultaneously and V acting only. The design resistances result from the maximum absolute bending moments of the two load cases at the 5 different locations:

$$r_{d,1} = 3.49 \tag{20}$$
$$r_{d,2} = 1.35 \tag{21}$$
$$r_{d,3} = 2.03 \tag{22}$$
$$r_{d,4} = 3.84 \tag{23}$$
$$r_{d,5} = 4.84 \tag{24}$$

Choosing the partial safety factor of the resistance as $\gamma_M = 1.3$, the characteristic resistances are calculated as $r_{k,i} = \gamma_M \cdot r_{d,i}$. We further choose the resistances to be log-normally distributed with $\text{c.o.v.}[R_i] = 0.1$ and the characteristic values to be defined via the 5[%] quantiles. This results in the following distributions of the resistances R_i:

$$R_1 \sim \mathcal{LN} \qquad \mathrm{E}[R_1] = 5.37 \qquad \text{c.o.v.}[R_1] = 0.1 \tag{25}$$
$$R_2 \sim \mathcal{LN} \qquad \mathrm{E}[R_2] = 2.08 \qquad \text{c.o.v.}[R_2] = 0.1 \tag{26}$$
$$R_3 \sim \mathcal{LN} \qquad \mathrm{E}[R_3] = 3.12 \qquad \text{c.o.v.}[R_3] = 0.1 \tag{27}$$
$$R_4 \sim \mathcal{LN} \qquad \mathrm{E}[R_4] = 5.93 \qquad \text{c.o.v.}[R_4] = 0.1 \tag{28}$$
$$R_5 \sim \mathcal{LN} \qquad \mathrm{E}[R_5] = 7.45 \qquad \text{c.o.v.}[R_5] = 0.1 \tag{29}$$

We further choose the resistances to be equicorrelated with a correlation coefficient of $\rho_{R_i,R_j} = 0.3$

4.2. Reliability analysis and link to the Daniels system

All minimal cut sets of bending failures at the 5 different locations, which lead to a kinematic system, define the system failure modes. We consider the five failure modes shown in Figure 5.

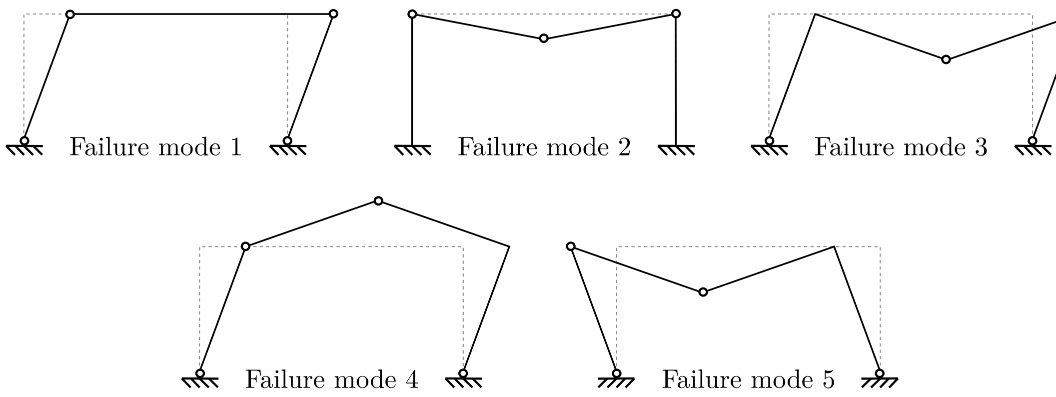

Figure 5: Considered failure modes of the frame.

Each of the failure modes is related to a Daniels system with similar reliability behavior. Hence, the system reliability is described by a series system of Daniels systems. How the system reliability is calculated and how the link between the frame and the Daniels system can be established, strongly depends on the material of the frame. We investigate two cases: Ideal plastic and ideal brittle material behavior. If the material behavior is semi-plastic, a link to the Daniels system can also be established; however, the load effects per damage state of the frame are not as straightforward to calculate anymore, and yield hinge theory is required.

4.2.1. Ideal plastic material behavior

Utilizing the principle of virtual work, a limit state function per failure mode can be derived. We demonstrate this in the following for the third failure mechanism. Figure 6 shows the kinematics of the third failure mode. From this the outer virtual work δW_O and the inner virtual work δW_I can be calculated as:

$$\delta W_O = H \cdot 5 \cdot \delta\varphi + V \cdot 5 \cdot \delta\varphi \tag{30}$$

$$\delta W_I = R_1 \cdot \delta\varphi + R_3 \cdot 2 \cdot \delta\varphi + R_4 \cdot 2 \cdot \delta\varphi + R_5 \cdot \delta\varphi \tag{31}$$

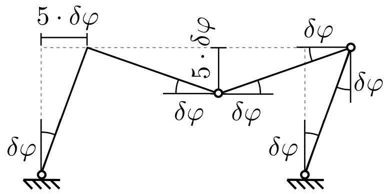

Figure 6: Kinematics of the third failure mechanism.

The inner virtual work represents the resistance of the system against a failure mode and the outer virtual work represents the load acting in the direction of a failure mode. From a statics point of view, R_i represents the actually acting resistance moment and the inner and outer virtual work have to be equal, otherwise, the system is kinematic. Reinterpreting R_i as the potential maximum bending moment resistance at location i, the following LSF regarding the third failure mechanism and the simulations load case are established:

$$g_3 = R_1 + 2 \cdot R_3 + 2 \cdot R_4 + R_5 - 5 \cdot H - 5 \cdot V \tag{32}$$

The frame fails in the third failure mode if and only if g_3 is negative.

Remark: The third failure mode cannot only be caused by the combined load case of $V \oplus H$ but also by the load case of V acting only. This would erase the term $-5 \cdot H$ in the limit-state function. The respective failure domain is a subset of the failure domain defined via the combined load case (assuming positive H). Therefore, we do not consider this limit-state function.

Analogously, the remaining failure modes lead to the following LSFs:

$$g_1 = R_1 + R_2 + R_4 + R_5 - 5 \cdot H \tag{33}$$

$$g_2 = R_2 + R_3 + R_4 - 5 \cdot V \tag{34}$$

$$g_4 = R_1 + 2 \cdot R_2 + 2 \cdot R_3 + R_5 + 5 \cdot V - 5 \cdot H \tag{35}$$

$$g_5 = R_1 + 2 \cdot R_2 + 2 \cdot R_3 + R_5 - 5 \cdot V \tag{36}$$

Note, that the contribution of V in the fourth failure mode is positive, therefore, V counteracts the failure mode.

System failure occurs if at least one failure mode occurs; hence, the system reliability is determined from a series system composed of the five LSFs. We approximate the reliability indices per failure mode and load case with the first-order reliability method (FORM) [14, 15]

$$\beta_{1,plast} = 4.66 \tag{37}$$
$$\beta_{2,plast} = 5.01 \tag{38}$$
$$\beta_{3,plast} = 5.28 \tag{39}$$
$$\beta_{4,plast} = 5.73 \tag{40}$$
$$\beta_{5,plast} = 7.64 \tag{41}$$

and the system reliability with FORM for series systems [16]:

$$\beta_{Sys,plast} = 4.62 \tag{42}$$

In the case of ideal plastic material behavior, it is possible to deduce a series system of Daniels systems which is – from a reliability point of view – equivalent to the frame. The series system of Daniels systems can be established as follows: The LSFs of Equations 32-36 do not only represent the failure mechanisms of the frame, but all failure mechanisms of any structural system which have the same inner and outer virtual work. This means, a Daniels system with the same inner and outer virtual is – from a reliability point of view – equivalent to a failure mode of the frame. It is always possible to find such a Daniels system. Figure 7 shows the resulting Daniels systems.

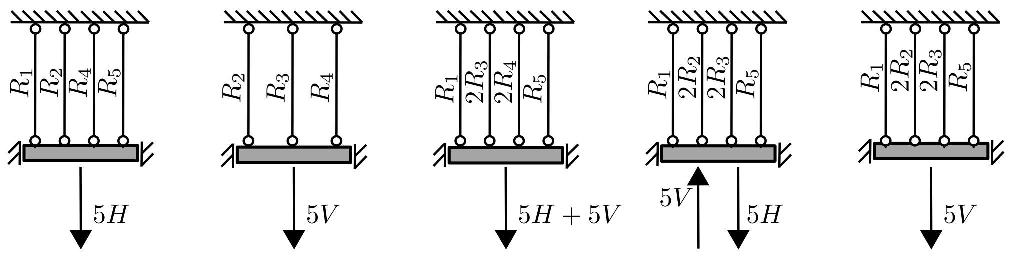

Figure 7: Daniels systems which are equivalent to the five failure modes of the example frame.

4.2.2. Ideal brittle material behavior

We carry out the reliability analysis via Monte Carlo simulation (MCS) [17, 18]. For each sample of the loads, we perform a structural analysis with respect to both load cases. If a resistance sample at one of the five locations of the frame is lower than the bending moment – caused by one of the load cases – we add a hinge at this location.[1] If multiple resistance samples are lower than the respective bending moments, we add the hinge at

[1] It may seem unrealistic to only add a hinge in case of brittle failure, since brittle failure typically does not allow transmission of normal forces and shear forces anymore; however, in the example setup we assumed only bending failure to be possible. In principle, the inclusion of normal force failure or shear force failure is possible; however, an example with higher amounts of redundancy would be needed to show meaningful calculations.

the location where the difference between resistance and bending moment is the greatest, whereby we favor cases that occur in the load case of the permanent load only. If a hinge is added, we again perform a structural analysis of the modified version of the frame potentially adding another joint. We iterate this procedure until the frame can either resit the loads or fails (becomes kinematic). The resulting estimate of the system reliability index is:

$$\beta_{Sys,brit} = 3.20 \tag{43}$$

Classifying the different forms of kinematics we estimate the reliability indices per failure mode as:

$$\beta_{1,brit} = 3.99 \tag{44}$$

$$\beta_{2,brit} = 3.22 \tag{45}$$

$$\beta_{3,brit} = \infty \tag{46}$$

$$\beta_{4,brit} = 4.74 \tag{47}$$

$$\beta_{5,brit} = 4.56 \tag{48}$$

In the case of ideal brittle material behavior, it is not possible to establish a one-to-one equivalents to the Daniels system as it is the case for the ideal plastic material behavior. The main reason for this is the difference in the load redistribution after one or more elements/members fail. In the case of the Daniels system, the relationship between member loads and resistances is equal for all non failed members (see Section A.6). This is not the case for the example frame; however, the standard partial safety factor design leads to member stiffnesses that are proportional to the load effects. This corresponds to the load distribution property of the extended Daniels system of Equation 74.

There are two main reasons which cause member stiffnesses that are non-proportional to the load effects:

- The consideration of multiple load cases: If the stiffnesses per member result from the maximum design stiffness of different load cases, member stiffnesses may not be proportional to the load effects caused by one of the load cases. However, this non-proportionality is not critical since it leads to an increase in the stiffness compared to the cases where only one load case is considered; hence, it increases the reliability.

- The redistribution of the load after one or multiple members fail: If a member of a structural system fails this may change the load flow of the structure fundamentally. The load effects change while the stiffnesses of the undamaged members remain the same, leading to a non-proportionality. This non-proportional redistribution of the load is critical, since the stiffness of some members may now be lower than the stiffness which would result from a partial safety factor design of an altered structure including the member failure.

In the following, we derive Daniels systems for the case of proportional and non-proportional load-sharing which are – from a reliability point of view – equivalent to the example frame. However, this equivalent only holds for the respective load case and damage state of the frame. In the case of a different load case or after failure of one element occurs, the characteristics of the equivalent Daniels system change.

- **Non-equal load-sharing proportional to the member stiffnesses**
  This case occurs if the frame is not damaged yet and only one of the load cases is considered for both: The partial safety factor design and the reliability assessment. We exemplarily illustrate the case of V acting only. The design resistances following the partial safety concept lead to resistances that are proportional to the load effects (the resistance can be derived analogously to Section 4.1). This proportionality directly defines the equivalent Daniels system (see Figure 8).

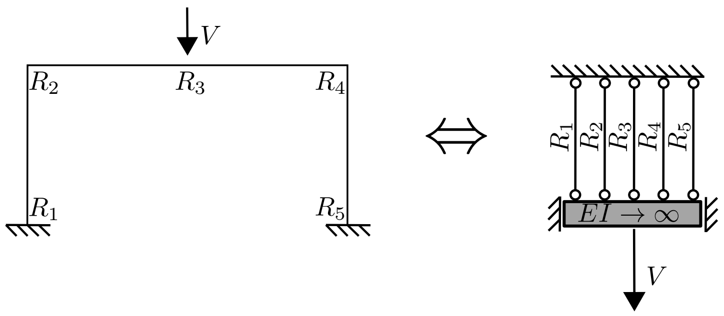

Figure 8: Undamaged frame loaded by V only and Daniels system representing the equivalent system behavior until element failure occurs. R_1-R_5 are determined following the partial safety concept considering only the load case of V acting alone.

- **Non-equal load-sharing non-proportional to the member stiffnesses**
  We exemplarily illustrate 3 cases:

  – Again the previous case (only V considered for both, the partial safety factor design and the reliability assessment), but with failure at location 2. The failure adds a joint to the frame which leads to different load effects at each location. These load effects are not proportional to the resistances anymore. To represent this, we modify the horizontal bar of the Daniels system: The bar is free to rotate and fixed against translations at one end (see Figure 9). By choosing the distance δl_i of the members to the fixed end of the horizontal bar for each member, the non-proportional load distribution can be represented. δl_i is calculated as:

$$\delta l_i = \left| \frac{M_{i,\Lambda_2}(V = v_d)}{M_i(V = v_d)} \right| \tag{49}$$

  where $M_i(V = v_d)$ and $M_{i,\Lambda_2}(V = v_d)$ are the inner moment at location i caused by the vertical design load v_d, if no failure or failure at location 2 is present.

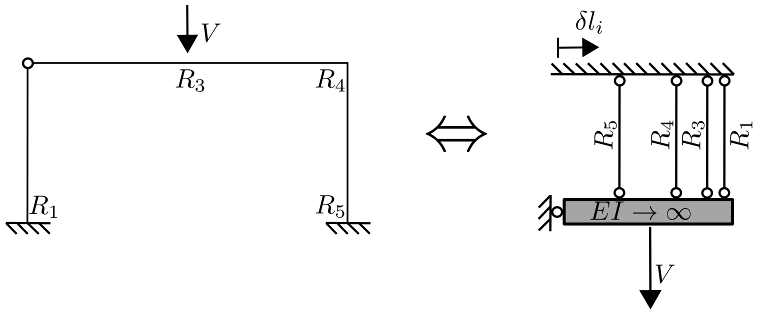

Figure 9: Frame with failure at location 2 loaded by V only and Daniels system representing the equivalent system behavior until another element failure occurs. R_1 and R_3-R_5 are determined with respect to the undamaged frame following the partial safety concept considering only the load case of V acting alone.

- The undamaged frame, where the load cases V and $V \oplus H$ are considered within the partial safety factor design but only V is taken into account within the reliability assessment:

  The Daniels system of Figure 9 can be derived similar to the previous case. δl_i is calculated as

$$\delta l_i = \left| \frac{M_i(V = v_d)}{\max\left\{M_i(V = v_d, h = h_d), M_i(V = v_d)\right\}} \right| \tag{50}$$

  where $M_i(V = v_d, H = h_d)$ is the inner moment of the undamaged frame at location i caused by the vertical and horizontal design load v_d and h_d.

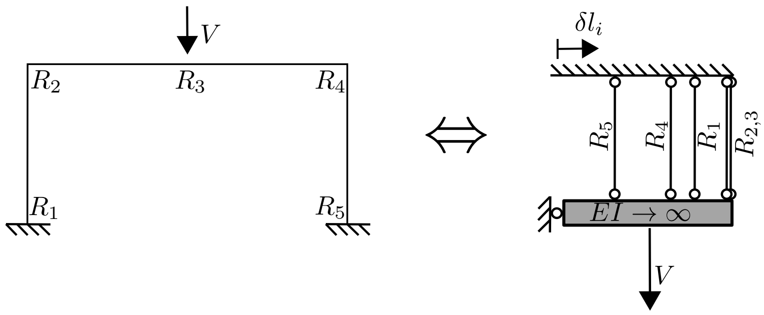

Figure 10: Undamaged frame loaded by V only and Daniels system representing the equivalent system behavior until element failure occurs. R_1-R_5 are determined following the partial safety concept considering both load cases V and $V \oplus H$.

- The undamaged frame, where the load cases V and $V \oplus H$ are considered within the partial safety factor design but only $V \oplus H$ is taken into account within the reliability assessment:

  Because two loads are acting in two different directions, the Daniels system needs to be modeled with two horizontal bars. The members are not directly attached to these horizontal bars anymore, but infinitely stiff connections transfer the load effects accordingly (see Figure 11). $\delta l_{i,V}$ and $\delta l_{i,H}$ are calculated as

$$\delta l_{i,V} = \left| \frac{M_i(V = v_d)}{\max\left\{M_i(V = v_d, h = h_d), M_i(V = v_d)\right\}} \right| \quad (51)$$

$$\delta l_{i,H} = \left| \frac{M_i(H = h_d)}{\max\left\{M_i(V = v_d, h = h_d), M_i(V = v_d)\right\}} \right| \quad (52)$$

Some of the force effects do not induce tension but compression on the members of the Daniels system (modeled with the help of a rocker). This is because the inner moments cause by V and H at a specific location of the frame have opposing signs. Consequently, the members of the Daniels system cannot only fail due to tension but also due to compression. If the bending stiffness at a location of the frame is independent of the bending direction (e.g. for symmetric cross-sections), the resistance against compression of the corresponding member of the Daniels system is equally distributed and fully correlated to their respective resistance against tension. If the bending stiffnesses differ with respect to the bending direction, the resistance against tension and against compression of the corresponding member of the Daniels system also differ accordingly (however, are still fully correlated).

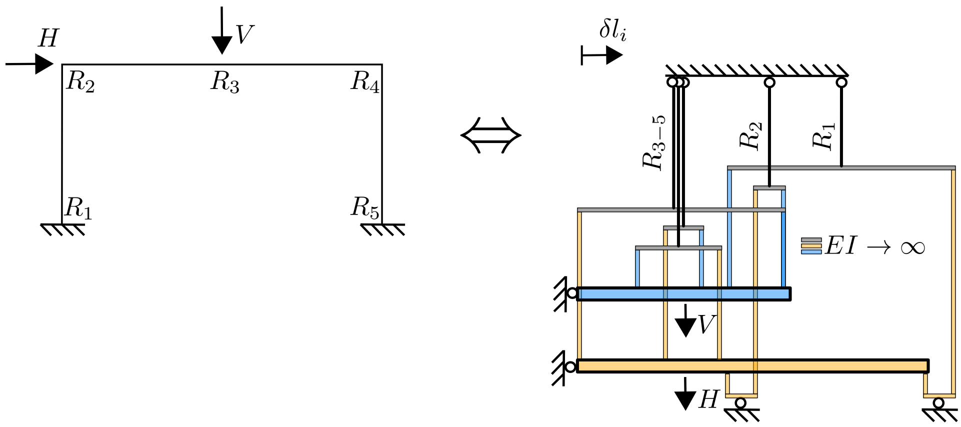

Figure 11: Undamaged frame loaded by $V \oplus H$ and Daniels system representing the equivalent system behavior until element failure occurs. R_1-R_5 are determined following the partial safety concept considering both load cases.

Note that, the equivalents of the frame and the above derived Daniels systems only hold, since the relationship between loads and load effects is linear. If a structure behaves nonlinearly, the shape of the horizontal bar needs to be adjusted (e.g. parabolic in case of quadratic nonlinearities).

4.3. Application of the additional partial safety factor

We apply γ_{Sys} to the example frame structure. The frame is 3 times statically overdetermined, therefore, 4 element failures make the frame kinematic ($n = 4$). The coefficient of variation of the member resistances is 0.1 and the members are equicorrelated with a correlation coefficient of 0.3; thus, the values of γ_{Sys} can be taken from the fourth column of Table 1. In the ideal plastic case, γ_{Sys} is 1.05. In the ideal brittle case, γ_{Sys} is 0.95.

We redesign the frame including γ_{Sys}. The distributions of the member resistances R_{1-5} are determined analogously to Section 4.1; whereby, $\gamma_M \cdot \gamma_{Sys}$ is applied to the characteristic resistances. The system reliability is calculated as in Section 4.2. The resulting system reliability indices are:

$$\beta_{Sys,plast,\gamma_{Sys}} = 4.43 \qquad \beta_{Sys,brittle,\gamma_{Sys}} = 3.45 \tag{53}$$

The previously calculated system reliabilities without the application of γ_{Sys} are $\beta_{Sys,plast} =$ 4.62 and $\beta_{Sys,brittle} = 3.20$ (see Equation 42 and 43). Hence, both reliability indices are closer to the target reliability index of 4.3.

5. Conclusion

We introduced an additional partial safety factor γ_{Sys} to take redundancy effect into account without leaving the framework of the PSF concept. It was derived by means of a generalized Daniels system. γ_{Sys} leads to a homogenization of the safety level. Due to necessary simplifications, the homogenization is not perfect; however, a step in the right direction.

γ_{Sys} depends on the static overdetermination of the system, its material behavior, the coefficient of variation of the material strength and the correlation of the involved element failure mechanisms. The last two quantities may not be given within an PSF design and have to be determined separately. If γ_{Sys} would be included within a structural code, recomandation of these quantities for different structural systems need to be derived.

The majority of values of γ_{Sys} is close to 1. This is a reassuring result, as it shows that the majority of current PSF designs are not very far from the target reliability due to system effects. Neither material wastage due to overdesign nor unsafe structures usually occur. Hence, the application of γ_{Sys} may not be worth the effort in most cases. Exceptions may be larger or critical structural systems (e.g. bridges) or structural systems which are build multiple times (e.g. prefabricated houses).

In some cases γ_{Sys} has rather low values up to 0.75 (high static overdetermination, high plasticity of the material, high coefficient of variation of the material strength and low correlation between element failure mechanism). In these cases a high saving potential of resources could be exploit; however, the application of γ_{Sys} is not needed due to safety issues. In other cases γ_{Sys} reaches values up to 1.10 (high static overdetermination, low plasticity of the material, low coefficient of variation of the material strength and high correlation between element failure mechanism). In these cases the application of γ_{Sys} is recommended to ensure sufficient safety of the structure.

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A. Extended Daniels system

In this appendix we give detailed derivations for the extended version of the Daniels system. For each extension we perform numerical studies. Moreover, we give a review on existing extension and relate the extensions to the ideal series and the ideal parallel system.

A.1. Extensions in literature

The finding of Daniels that the system resistance is asymptotically normally distributed was improved in [19] and extended, such that correlation among members [10, 20], more general force-deformation curves [21], local load-sharing (stress concentrations of members which are next to failed members) [22–24] and time-dependent deterioration [25] can be considered. Different and not necessarily Gaussian asymptotic behavior was deduced for these different extensions. However, the convergence to the asymptotic result is slow; this holds for the original formulation as well as for all extensions [26]. This means that the limiting distribution of the system resistance is only suitable for systems with a large number of members n.

If the number of members n is smaller, a more exact evaluation of the system resistance is necessary. In this regard, we are not aware of any research other than the works of Gollwitzer, Hohenbichler, and Rackwitz. They utilized an order-statistics approach and reinterpreted system failure as the intersection of failure events [27]. The probability of this intersection can be approximated via FORM for parallel systems [16]. The approximation error of this approach is not negligible but represents a major improvement over the asymptotic approach. This approach allows the relaxation of some of Daniels' original assumptions. Gollwitzer and Rackwitz utilized this to carry out numerous numerical studies [11, 28]. In fact, they investigated three out of the four subsequent extensions to the Daniels system, namely, load modeling, material modeling, and correlation. However, our approaches differ fundamentally: They are not based on FORM for parallel systems, but either analytical, based on standard FORM or sampling methods. Our material model is simpler than the one of Gollwitzer [11] which is based on material model for timber with multiple parameters to calibrate. This may make our model less accurate in some cases, however, more general and applicable for other cases. Moreover, our approach to include correlation is more general and adaptable for all kinds of correlation.

A.2. Relationship to the ideal parallel and series system

Figure 12 shows a mechanical interpretation of an ideal parallel system and an ideal series system. System failure of the ideal parallel system is defined as the state where all members fail; however, it is hard to find a structural system for which this definition of failure is meaningful. The ideal series system fails if at least one of the members fails. The respective probabilities of system failure are

$$\Pr(F_{Sys,n;\text{Parallel}}) = \left[F_R\left(\frac{s}{n}\right)\right]^n \tag{54}$$

$$\Pr(F_{Sys,n;\text{Series}}) = 1 - \left[1 - F_R\left(s\right)\right]^n \tag{55}$$

where s is the load, n is the number of i.i.d. members and F_R is the CDF of the member resistances.

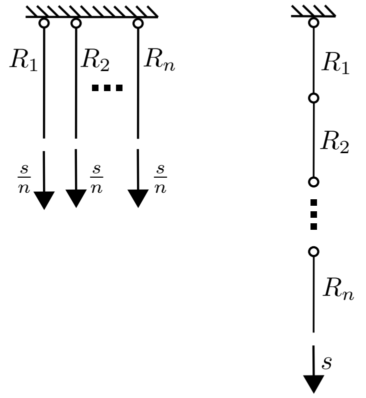

Figure 12: Mechanical representation of an ideal parallel system (left) and an ideal series system (right).

The ideal parallel system and the ideal series system represent the two extreme cases of system behavior. They, therefore, provide an upper and lower bound to the probability of system failure of the Daniels system:

$$\Pr(F_{Sys,n;\text{Parallel}}) \leq \Pr(F_{Sys,n;\text{Daniels}}) \leq \Pr(F_{Sys,n;\text{Series}}) \tag{56}$$

A.3. Probabilistic load modeling

Let $l(\mathbf{S})$ be a function modeling the load of the Daniels System. $\mathbf{S}$ is a vector of load phenomena (e.g., wind and snow) and the function l represents their combined effect. Assuming independence of the resistances and the load, the probability of system failure can be calculated by application of the total probability theorem:

$$\Pr(F_{Sys,n}) = \int_{\Omega_{\mathbf{S}}} F_{R_{Sys,n}}\left(l(\mathbf{S})\right) \cdot f_{\mathbf{S}}(\mathbf{s})\mathrm{d}\mathbf{s} \tag{57}$$

where $\Omega_{\mathbf{S}}$ is the sample space of $\mathbf{S}$ and $f_{\mathbf{S}}$ is the joint PDF of $\mathbf{S}$.

As an alternative to direct numerical evaluation of the integral of Equation 57, the FORM can be used. Following [29, 30], we formulate the limit state function (LSF) of a single component reliability problem:

$$g(P,\mathbf{S}) = P - \Pr(F_{Sys,n}) \tag{58}$$

where P is a random variable with standard uniform distribution and $\Pr(F_{Sys,n})$ can be calculated via the recursive formula of Equation 3. Transformation to standard normal space results in

$$G(u_P,\mathbf{u_S}) = u_P - \Phi^{-1}\left(F_{R_{Sys,n}}(l(T_{U2X}\left(\mathbf{u_S}\right)))\right) \tag{59}$$

where U_P is standard normally distributed, $\mathbf{u_S}$ is multivariate standard normally distributed and T_{U2X} is the transformation from the standard normal space to the original space. $G(u_P, \mathbf{u_S})$ is suitable for application within FORM. [2] [3]

A.3.1. Numerical investigations of the probabilistic load modeling

We perform numerical investigations with a log-normally distributed resistance ($\mathrm{E}[R] = 1$ and $\mathrm{c.o.v.}[R] = 0.1$) and a one-dimensional load following a Gumbel distribution ($\mathrm{c.o.v.}[S] = 0.2$). $\mathrm{E}[S]$ is chosen such that a target reliability index $\beta_T = 4.3$ is achieved in case of a Daniels system with only one member.

Figure 13 illustrates how the reliability index changes in the case of $n > 1$ for different values of $\mathrm{c.o.v.}[S]$. With increasing $\mathrm{c.o.v.}[S]$, the reliability index of the Daniels system is less sensitive to n. Changes of $\mathrm{c.o.v.}[S]$ in low ranges (0.01-0.05) have a stronger effect on the reliability index of the Daniels system. Values of $\mathrm{c.o.v.}[S]$ in the range of (0.1-0.5) lead to similar reliability curves. In this range, the reliability of the Daniels system keeps decreasing with increasing n.

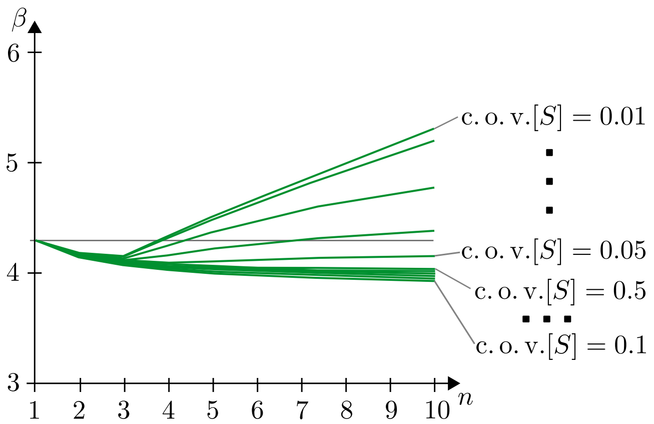

Figure 13: Reliability index of the Daniels system with 1 to 10 members with different c. o. v. of S of 0.01, 0.02, ..., 0.05 and 0.1, 0.2, ..., 0.5.

[2] Alternatively, the LSF can be formulated as

$$g(R_{Sys,n}, \mathbf{S}) = R_{Sys,n} - l(\mathbf{S}) \tag{60}$$

and transformed to the standard normal space as

$$G(u_R, \mathbf{u_S}) = F^{-1}_{R_{Sys,n}}\left(\Phi(u_R)\right) - l\left(T_{U2X}\left(\mathbf{u_S}\right)\right) \tag{61}$$

This LSF is also suitable for application within FORM, however, it requires the numerical evaluation of $F^{-1}_{R_{Sys,n}}$, which is not available in closed form.

[3] Note that FORM should not be applied if the system resistance is formulated as in Equation 1. The LSF would be:

$$g(R_{Sys,n}, \mathbf{S}) = \max_{i=1}^{n}\left\{(n-i+1)\cdot R_{(i)}\right\} - l(\mathbf{S}) \tag{62}$$

Applying FORM to this formulation of the LSF leads to incorrect results because the corresponding limit-state surface is highly non-linear.

A.4. Material models

The stress-strain relationship of the considered material model is shown in Figure 14. It is linear until a maximum stress σ_{max} is reached. If the strain increases further, the stress drops to a constant residual stress σ_{plast} [31].

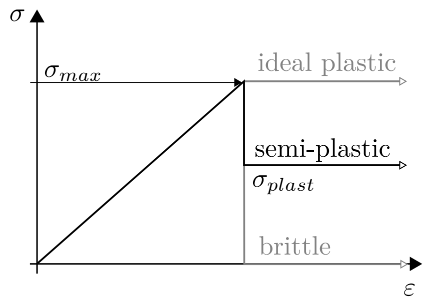

Figure 14: Stress-strain relationship for new material models.

We define σ_{plast} in two different ways. Either as a deterministic value:

$$\text{Material model 1: } \sigma_{plast,1} = f_{res} \cdot \mathrm{E}[\sigma_{max}] \qquad f_{res} \in [0;1] \tag{63}$$

or probabilistically depending on σ_{max}:

$$\text{Material model 2: } \sigma_{plast,2} = f_{res} \cdot \sigma_{max} \qquad f_{res} \in [0;1] \tag{64}$$

f_{res} is a factor that quantifies the residual strength. If $f_{res} = 0$, both material models represent brittle material behavior. If $f_{res} = 1$, the second material model represents ideal plastic material behavior.

The first material model can be "unrealistic" in the sense that $\sigma_{plast,1}$ can be greater than small instantiations of σ_{max} so that the stress-strain relationship would have an upward jump. However, this model has the advantage that Daniels' formula (Equation 3) can be adapted such that the probability of failure can still be calculated analytically. This is not the case for the second material model.

In case of material model 1, the plastic resistance of the failed members can be reinterpreted as an additional negative load. Then the recursive formula to evaluate the CDF of the system resistance (Equation 3) can be adapted to calculate the probability of system failure under a deterministic load s as

$$\begin{aligned} F_{R_{Sys,m}}(s) =& (-1)^{m+1} \cdot F_R^m\left(\frac{s-(n-m)\cdot\sigma_{plast,1}}{m}\right) \\ & - \sum_{j=1}^{m-1}\left[(-1)^j \cdot \binom{m}{j} \cdot F_R^j\left(\frac{s-(n-m)\cdot\sigma_{plast,1}}{m}\right) \cdot \right. \\ & \left. F_{R_{Sys,m-j}}(s-(n-m)\cdot\sigma_{plast,1})\right] \end{aligned} \tag{65}$$

where m is an auxiliary variable of the recursion. The recursion has to be conducted for $m = 1, \ldots, n$.

Equation 65 only calculates the probability of system failure if the plastic resistance of the system is smaller than the load $\sigma_{plast,1} \cdot n < s$. If $\sigma_{plast,1} \cdot n > s$, the probability of system failure $\Pr(F_{Sys,n})$ is zero.

In case of material model 2, the probability of failure can be calculated via the following n-fold parameter integral:

$$\Pr(F_{Sys,n}) = n! \cdot \int_0^{\frac{s}{n}} \int_{r_n}^{\frac{s}{n-1} - f_{res} \cdot r_n} \cdots \int_{r_2}^{s - \sum_{i=1}^{n-1} f_{res} \cdot r_i} f_R(r_n) \cdot f_R(r_{n-1}) \ldots f_R(r_1) \, \mathrm{d}r_1 \ldots \mathrm{d}r_{n-1} \, \mathrm{d}r_n \tag{66}$$

For larger n, it is not feasible to evaluate the integral with classic numerical integration methods. Furthermore, FORM is not suitable because of the shape of the failure domain. Instead MCS or advanced sampling-based methods, such as Subset Simulation (SuS) [32–34], can be applied to estimate $\Pr(F_{Sys,n})$. MCS is straightforward to implement and has guaranteed accuracy. However, for small $\Pr(F_{Sys,n})$, MCS is inefficient. SuS is an adaptive sampling method, which is suitable for small $\Pr(F_{Sys,n})$.[4]

In the special case of the full plastic material behavior (material model 2 with $f_{res} = 1$), the system resistance is the sum of the member resistances. The limit-state function is

$$g = \sum_{i=1}^{n} R_i - s \tag{69}$$

The corresponding limit-state surface is linear in the original space; hence, the application of FORM is suitable.

[4] MCS and SuS can also be combined with the analytic solutions provided in Equation 65. E.g. in the case of a Daniels system with two members of material 2, the probability of system failure can be calculated as

$$\Pr(F_{Sys,n}) = 2! \cdot \int_0^{\frac{s}{2}} \int_{r_2}^{s - f_{res} \cdot r_2} f_R(r_1) \cdot f_R(r_2) \mathrm{d}r_1 \mathrm{d}r_2 \tag{67}$$

This integral can be split as

$$\Pr(F_{Sys,n}) = 2! \cdot \left[\int_0^{\frac{s}{2}} \int_{r_2}^{s - f_{res} \cdot \frac{s}{2}} f_R(r_1) \cdot f_R(r_2) \mathrm{d}r_1 \mathrm{d}r_2 + \underbrace{\int_0^{\frac{s}{2}} \int_{s - r \cdot \frac{s}{2}}^{s - f_{res} \cdot r_2} f_R(r_1) \cdot f_R(r_2) \mathrm{d}r_1 \mathrm{d}r_2}_{=: I^+} \right] \tag{68}$$

The first integral of Equation 68 can be calculated exactly by applying Equation 65. Hence only the error in the estimation of I^+ with MCS or SuS remains. Since $I^+ < \Pr(F_{Sys,n})$, Equation 68 leads to a variance reduction of the estimator compared to a direct application of MCS or SuS to Equation 67. This approach can be extended to the case of a Daniels system with n members.

A.4.1. Numerical investigations with the alternative material model

We perform the numerical investigations with the same setup as in A.3.1. Figure 15 shows the reliability indices of Daniels systems with members modeled with material models 1 and 2 for different degrees of plasticity (f_{res} = 0, 0.2, 0.4, 0.6, 0.8, 1.0). For low values of f_{res}, the two material models lead to similar results. For larger f_{res}, the first material model leads to much larger reliability indices than the second material model. For this first material model, β eventually becomes infinite as $\sigma_{plast,1} \cdot n > s$. The reliability indices resulting from material model 2 are bounded by the ideal parallel system.

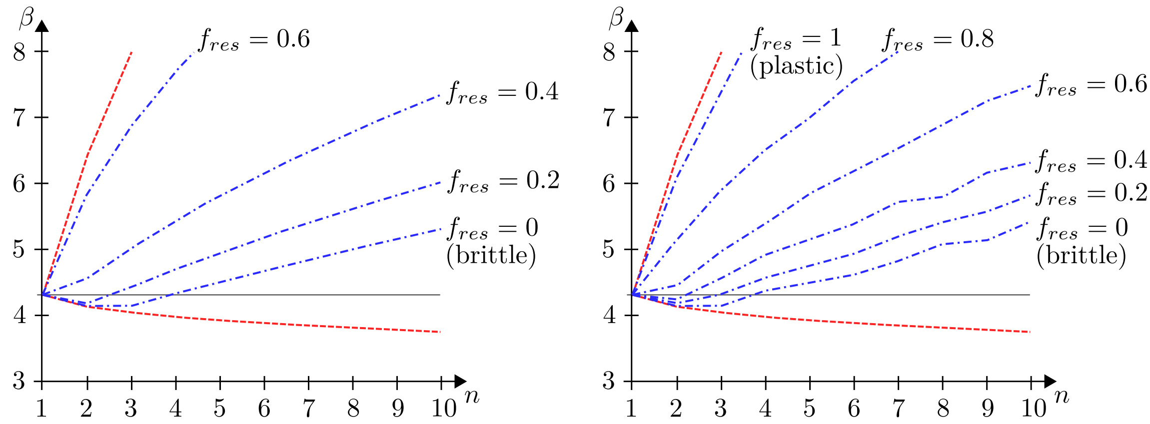

Figure 15: Reliability index of the Daniels system with members modeled with material model 1 (left) and model 2 (right) for different degrees of plasticity f_{res} (blue dash-dotted) and ideal series and parallel system (red dashed). The reliability indices associated with material model 2 were calculated with the SuS-implementation of [34] with an intermediate probability per level of 0.1 and 10^4 samples per level.

A.5. Correlation among members

We extend the Daniels system to include equicorrelated correlated members by means of a hierarchical model [6, 35, 36]. In the case of equicorrelation a hierarchical model with only only one hyperparameter α is required to represent the dependence structure (see Figure 16). The approach can be extended to unequal correlation among members, as discussed further below.

It is mathematically convenient to choose a hyperparameter α that follows a standard normal distribution. Additionally, n standard normal distributed auxiliary random variables Y_i are introduced. The Y_i are equicorrelated jointly normal with correlation coefficient ρ_Y, which follows from the correlation of the R_i transformed into the standard normal space [37, 38].

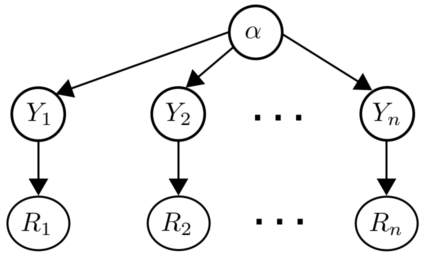

Figure 16: Hierarchical Bayesian network with hyperparameter α to model equicorrelation among member resistances.

The conditional CDF $F_{R|\alpha}$ is given as [35]

$$F_{R|\alpha}(x \mid \alpha) = \Phi\left(\frac{\Phi^{-1}\left(F_R(x)\right) - \sqrt{\rho_Y} \cdot \alpha}{\sqrt{1-\rho_Y}}\right) \tag{70}$$

The correlation model can be extended to resistances R_i with differing marginal distributions and varying mutual correlation coefficients. The only restriction is that the correlation matrix in standard normal space has to be of the Dunnett-Sobel class [6,39].

The probability of failure is calculated via the total probability theorem as

$$\Pr(F_{Sys,n}) = \int_{-\infty}^{\infty} \varphi(\alpha) \cdot \Pr(F_{Sys,n} \mid \alpha)\mathrm{d}\alpha \tag{71}$$

where φ is the standard normal PDF. $\Pr(F_{Sys,n} \mid \alpha)$ is calculated with the original formula of Daniels (Equation 3), whereby the CDF of the member resistances is $F_{R_i|\alpha}$ defined via Equation 70. The integral in Equation 71 can be evaluated numerically. Alternatively, $\Pr(F_{Sys,n})$ can be approximated by FORM. By analogy with Equation 58-59 FORM is applied to the following LSF:

$$G(U_P,\alpha) = U_P - \Phi^{-1}\left(F_{R_{Sys,n}|\alpha}(s)\right) \tag{72}$$

where U_P and α follow a standard normal distribution and $F_{R_{Sys,n}|\alpha}$ is the CDF of the system resistance following equation 3, whereby the CDF of the member resistances is $F_{R_i|\alpha}$ defined via Equation 70.

A.5.1. Numerical investigations of the correlation model

We perform the numerical investigations with the same setup as in A.3.1. Figure 17 shows the influence of the correlation between member resistances of the Daniels system with ideal plastic, ideal brittle, and semi-plastic material modeled with material model 2 with $f_{res} = 0.5$. With increasing correlation, an increase of the number of members has less effect on the reliability. If the members are fully correlated, the Daniels system degenerates to a single component system.

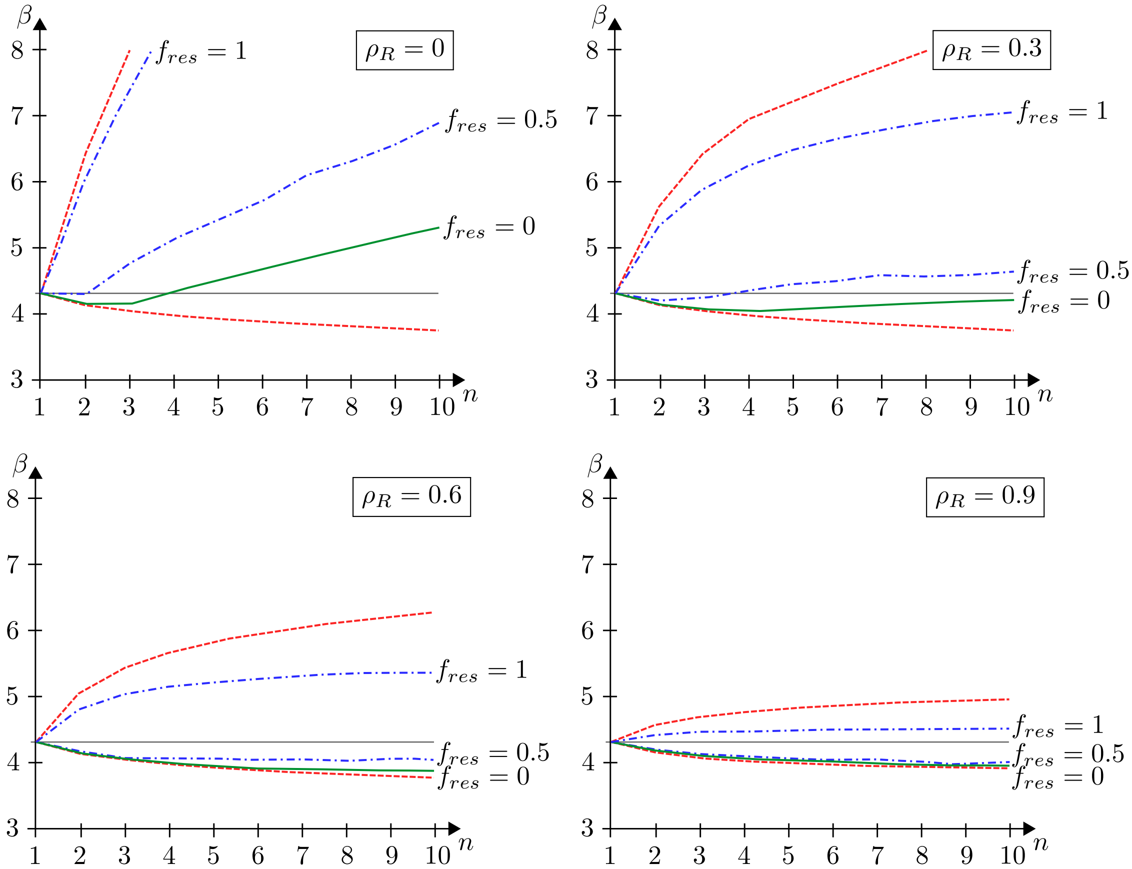

Figure 17: Reliability index of the ideal parallel and series system (dashed red) and the Daniels system with brittle ($f_{res} = 0$, green), ideal plastic and semi plastic material modeled with material model 2 with ($f_{res} = 1, 0.5$, dash-doted blue). The members are uncorrelated (top left), equicorrelated with a correlation coefficient of 0.3 (top right), 0.6 (bottom left) and 0.9 (bottom right).

A.6. Modified load-sharing properties among members

The original Daniels system has the following property of equal load-sharing among non-failed members:

$$\forall_{i,j\in\{1,\dots,n\}}\forall_{\Lambda\subseteq\{1,\dots,n\}\setminus\{i,j\}} : \; s_{i,\Lambda} = s_{j,\Lambda} \tag{73}$$

where $s_{i,\Lambda}$ $i \in \{1, \dots, n\}$ is the share of the total load s acting on the ith member of the Daniels system with failed members $\Lambda \subseteq \{1, \dots, n\} \setminus i$ (the undamaged Daniels system is represented via $\Lambda = \varnothing$). The equal load-sharing property is a consequence of two assumptions of the original Daniels system: First, the postulate of equal cross sections and Young's modulus, i.e., equal stiffnesses among all members. Second, the original Daniels system is modeled with a horizontal bar with infinite bending stiffness, which is blocked against rotation on both sides. Altering any of these properties leads to non-equal load-sharing.

Furthermore, all resistances of the original Daniels system follow the same distribution.

This makes the relationship between member loads and resistances equal for all members.

Figure 18 shows three modifications of the Daniels system leading to different load-sharing properties.

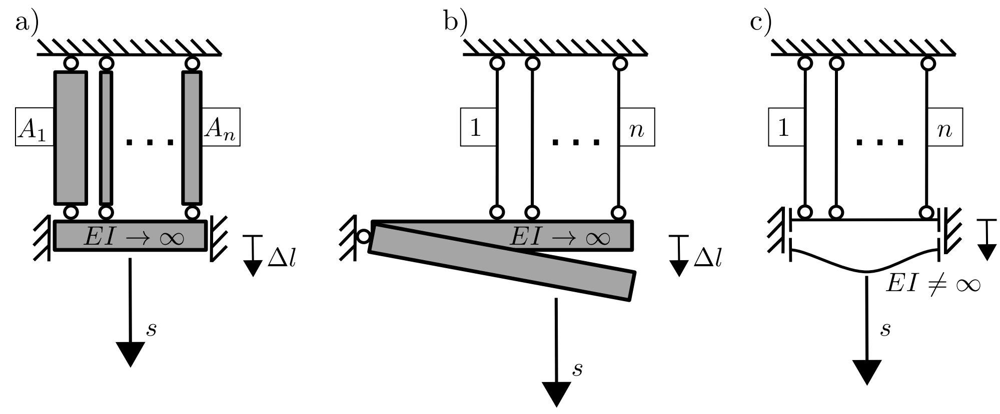

Figure 18: Modifications of the Daniels system which lead to different load-sharing properties.

Figure 18 a): This modified Daniels system has varying cross-section areas per member. For this modified Daniels system, Equation 73 does not hold anymore; however, another – slightly weaker – load-sharing property holds: The load is distributed proportional to the member stiffnesses:

$$\forall_{i,j \in \{1,\ldots,n\}} \forall_{\Lambda \subseteq \{1,\ldots,n\} \setminus \{i,j\}} : \frac{s_{i,\Lambda}}{A_i \cdot E} = \frac{s_{j,\Lambda}}{A_j \cdot E} \tag{74}$$

where E is the Young's modulus and A_i is the cross-section area of member i.

Figure 18 b): This modified Daniels system has a rotatable horizontal bar which is fixed at one end only. For this modified Daniels system, Equation 73 and 74 do not hold; however, another – again slightly weaker – load-sharing property holds: After the failure of member k, the load redistributes proportionally:

$$\forall_{i,j \in \{1,\ldots,n\}} \forall_{\Lambda \subseteq \{1,\ldots,n\} \setminus \{i,j\}} \forall_{k \in \{1,\ldots,n\} \setminus (\Lambda \cup \{i,j\})} : \frac{s_{i,\Lambda}}{s_{j,\Lambda}} = \frac{s_{i,\Lambda \cup k}}{s_{j,\Lambda \cup k}} \tag{75}$$

Figure 18 c): This modified Daniels system has a horizontal bar with a finite bending stiffness. For this modified Daniels system, none of the properties (Equation 73-75) hold.

The three above-mentioned load-sharing properties (Equation 73-75) are logically related to each other as follows: Eq.73 ⇒ Eq.74 ⇒ Eq.75; hence, they can be interpreted as three levels of load-sharing leading to three levels of redundancy.

In order to calculate the system reliability of modified Daniels systems, it is helpful to compare the failure domain of the original Daniels system and a modified Daniels system.

Figure 19 visualizes this for the case of three members. The failure domain of the original Daniels system is the union of 6 cubes whose edge lengths are all possible combinations of s, $\frac{s}{2}$, and $\frac{s}{3}$. The recursive formula from Daniels (Equation 3) makes use of the symmetric shape of the failure domain and the fact that the member resistances are i.i.d.. The failure domain of modified Daniels systems also consists of 6 cubes; however, these are not necessarily symmetric anymore and the member resistances are not necessarily identically distributed random variables.

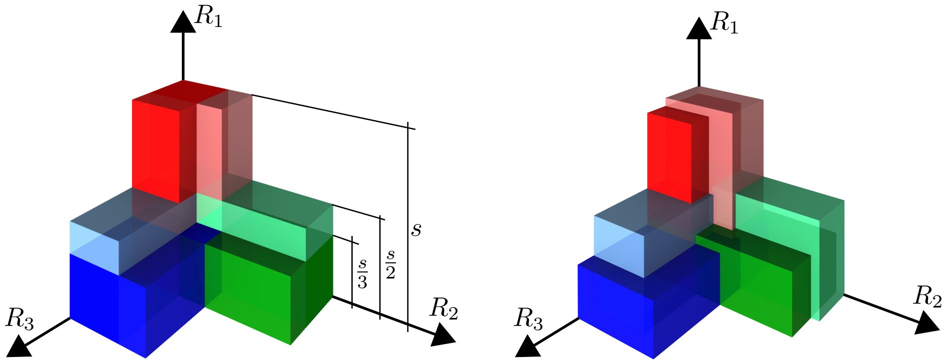

Figure 19: Failure domain of the original Daniels system (left) and a modified Daniels system (right) with 3 members.

In the general case of a modified Daniels system with n members, the failure domain is described by $n!$ hypercubes H_j ($j = 1, \ldots, n!$). Each hypercube describes one possible ordering of member failures leading to system failure. The edge lengths of a hypercube are equal to the load share $s_{i,\Lambda}$ where i is the next member to fail and Λ are members that have already failed. In the case of a proportional redistribution after member failure (Equation 75), the calculation of all $s_{i,\Lambda}$ is straightforward. If Equation 75 does not hold, the calculation of each $s_{i,\Lambda}$ requires a structural analysis.

In the following, we present two analytical approaches to calculate the probability of failure of modified Daniels systems as in Figure 18. The first approach is via the application of the principle of inclusion and exclusion. The second approach meshes the failure domain with disjoint hypercubes and sums up their respective probabilities. Both approaches are only reasonable for moderately small numbers of members n. For larger n, MCS [17, 18] or advanced sampling methods such as SuS [32–34] should be applied.

The first approach is baseed on the principle of inclusion and exclusion. The probability of system failure $F_{Sys,n}$ can be evaluated as

$$\Pr(F_{Sys,n}) = \sum_{k=1}^{n!} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,n!\} \\ |I|=k}} \Pr\left(\bigcap_{j\in I} H_j\right) \tag{76}$$

where the second summation is with respect to all possible index sets I of numbers from 1 to $n!$ and cardinality k.

With an increasing number of members n, the evaluation of Equation 76 becomes numerically infeasible even for moderate n. This is not only because the number of hypercubes

H_j grows with $\mathcal{O}(n!)$, but in particular since the number m-tuples to describe the intersections grows with $\mathcal{O}(n!!)$. The largest numerically reasonable number of elements for applying the principle of inclusion and exclusion is $n = 4$. In this case $4! = 24$ hypercubes H_j exist. This leads to $\binom{24}{2} = 276$ intersection pairs, $\binom{24}{3} = 2024$ intersection triplets etc. Reaching the maximum at $\binom{24}{12} = 2\,704\,156$ intersection 12-tuples. If the system consists of 5 members, $5! = 120$ hypercubes H_j exist. The number of intersection tuples is already 10^{36}.

A numerically preferable alternative to the principle of inclusion and exclusion is the following meshing approach. We divide the failure domain into non-overlapping hypercubes. Then the probability of system failure is calculated as the sum of the probabilities of all events defined via these hypercubes. One possibility to define the hypercubes and calculate their respective probability is to envelop the system failure domain with the hypercube $[0,s] \times [0,s] \times \cdots \times [0,s]$ and mesh it per direction ($\widehat{=}$member) i with the grid $s_{i,\Lambda}$ ($\widehat{=}$ load share of member i regarding system state with failed members Λ). If all $s_{i,\Lambda}$ for all Λ differ, the meshing defines a maximum of $n!^n$ sub-hypercubes h_j.

Summing over the probability of all sub-hypercubes within the failure domain $\Omega_{F_{Sys,n}}$ gives the probability of system failure

$$\Pr(F_{Sys,n}) = \sum_{h_j \subseteq \Omega_{F_{Sys,n}}} \Pr(h_j) \tag{77}$$

This approach has complexity $\mathcal{O}\left(n!^n\right)$, hence is more feasible than the application of the principle of inclusion and exclusion. The maximum numerically feasible number of members is in the order of $n \approx 10$.

A.6.1. Numerical investigations of modified load-sharing among members

In the following, we apply the meshing approach to the modified Daniels systems a) and b) of Figure 18. As for the original Daniels system, we assume equal Young's modulus per member, a deterministic load s (chosen such that the target reliability of $\beta_T = 4.3$ is met for $n = 1$), i.i.d ultimate member strength $\sigma_{max,i}$ and linear-elastic brittle material behavior.

Neither system satisfies the property of equal load-sharing among non-failed members (Equation 73). We introduce variables for both systems controlling the degree by which the systems deviate from the perfect load-sharing property of the original Daniels system and investigate its effect on the system reliability.

System a) of Figure 18
We assume the member strength to be log-normally distributed

$$\sigma_{max,i} \sim \mathcal{LN} \qquad \mathrm{E}[\sigma_{max,i}] = 1 \qquad \mathrm{c.\,o.\,v.}[\sigma_{max,i}] = 0.1 \tag{78}$$

and vary the cross-section areas linearly

$$A_i = \begin{cases} 1 & n = 1 \\ 1 - f_s \cdot \left(2 \cdot \frac{i-1}{n-1} - 1\right) & n \geq 2 \end{cases} \tag{79}$$

where n is the number of members of the modified Daniels system and $f_s \in [0,1)$ is a factor controlling the non-equality of the cross-section areas. The cross-section area of the first member is $A_1 = 1 - f_s$ and the cross-section area of the last member is $A_n = 1 + f_s$. The cross-section area of all other members is linearly interpolated between A_1 and A_n. If $f_s = 0$, the original Daniels system is obtained. The larger f_s, the more the cross-section areas differ, with a maximum inequality for $f_s = 1$.

Figure 20 illustrates the resulting system reliability indices for a Daniels system with $n = 5$ members. The reliability index is not significantly influenced by the factor f_s. We obtain similar results for other numbers of members n or different setups of the modified Daniels system (e.g. different c. o. v.$[\sigma_{max,i}]$ or semi-plastic material behavior). This indicates that in general, the reliability index may be not sensitive to non-equal load-sharing if the load-sharing property of Equation 74 is fulfilled (proportionality to the mean member resistances).

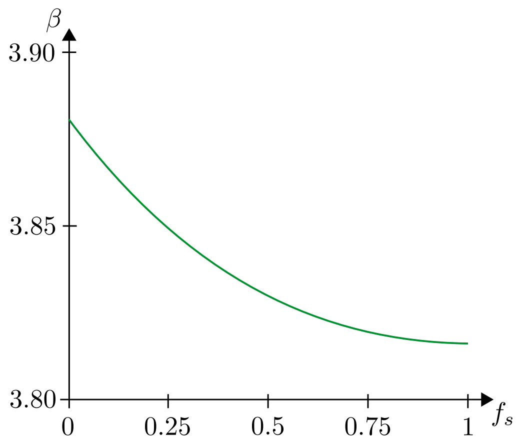

Figure 20: System reliability index of the modified Daniels system a) with $n = 5$ brittle members and different levels of non-equal load-sharing controlled via f_s (the original Daniels system corresponds to $f_s = 0$).

System b) of Figure 18
We assume the member strength to be log-normally distributed:

$$\sigma_{max,i} \sim \mathcal{LN} \qquad \mathrm{E}[\sigma_{max,i}] = 1 \qquad \text{c. o. v.}[\sigma_{max,i}] \in \{0.1, 0.2, 0.3\} \tag{80}$$

and all distances between members to be equal; therefore, the load is linearly distributed among members. The inequality of load-sharing is described via the ratio of the deformation of the first member to the last member $\frac{\Delta l_1}{\Delta l_n}$. If $\frac{\Delta l_1}{\Delta l_n} = 1$, the support of the horizontal bar is infinitely far away from the first member of the modified Daniels system b). This is equivalent to the original Daniels system. With decreasing $\frac{\Delta l_1}{\Delta l_n} < 1$, the support moves closer to the first member, and the load-sharing becomes increasingly unequal. The maximum inequality in load-sharing is reached for $\frac{\Delta l_1}{\Delta l_n} = 0$.

Figure 21 shows the resulting system reliability index. The system reliability index first decreases with the increasing number of members, reaches a minimum and then increases. Figure 22 shows that the coefficient of variation of the member strength has a significant influence on how much $\frac{\Delta l_1}{\Delta l_n}$ influences the resulting system reliability indices.

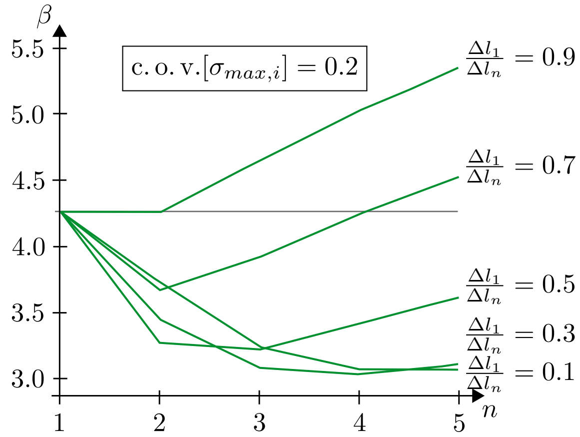

Figure 21: System reliability index of the modified Daniels system b) with n brittle members and different levels of non-equal load-sharing controlled via $\frac{\Delta l_1}{\Delta l_n}$.

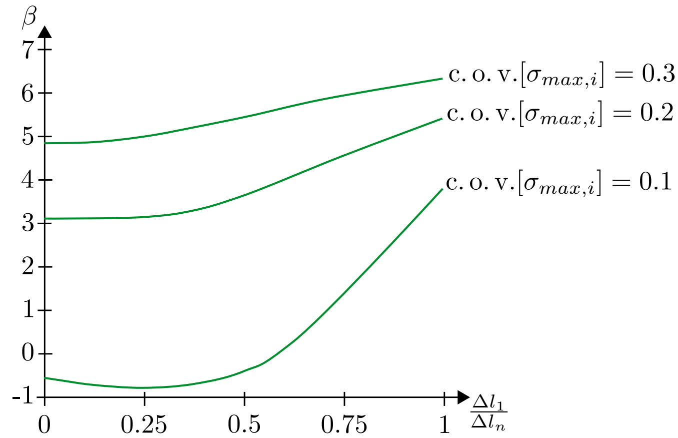

Figure 22: System reliability index of the modified Daniels system b) with $n = 5$ brittle members with varying c. o. v.[$\sigma_{max,i}$] $\in \{0.1, 0.2, 0.3\}$ and different levels of non-equal load-sharing controlled via $\frac{\Delta l_1}{\Delta l_n}$.