# Reliability analysis examples

Engineering Risk Analysis Group, Technische Universität München.
Arcisstr. 21, 80333 Munich, Germany.

May 18, 2018

## 1 Introduction

In the context of reliability analysis and rare event estimation, the performance of the system under consideration can be described by a *limit state function* (LSF) $g : \boldsymbol{\Theta} \to \mathbb{R}$. The failure hypersurface defined by $g(\boldsymbol{\theta}) = 0$ splits the space of uncertain variables $\boldsymbol{\Theta}$ into the safe domain $\mathcal{S} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} : g(\boldsymbol{\theta}) > 0\}$ and the failure domain $\mathcal{F} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} : g(\boldsymbol{\theta}) \leq 0\}$. The LSF can include one or several distinct modes of failure. The probability of occurrence of $\mathcal{F} \subseteq \boldsymbol{\Theta}$, called the *probability of failure*, is defined by

$$\mathrm{P}(\mathcal{F}) = \int_{\boldsymbol{\Theta}} \mathbb{I}_{\mathcal{F}}[\boldsymbol{\theta}] f(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{1}$$

where, $f(\boldsymbol{\theta})$ is the joint probability density function (PDF) of the model parameters and $\mathbb{I}[\cdot]$ is the indicator function, which takes the values $\mathbb{I}_{\mathcal{F}}[\boldsymbol{\theta}] = 1$ when $\boldsymbol{\theta} \in \mathcal{F}$, and $\mathbb{I}_{\mathcal{F}}[\boldsymbol{\theta}] = 0$ otherwise. A special challenge involves the analysis of *rare events*, that is, when Eq. (1) represents the solution of a high dimensional integral for which $\mathrm{P}(\mathcal{F})$ is very small.

In the following, we describe 6 examples that are commonly used as benchmarks in reliability analysis. These problems are solved in MATLAB® and in python™ 3 using the following approaches:

- First order reliability method (FORM) with the Hasofer-Lind-Rackwitz-Fiessler (HLRF) algorithm [12, 13]:
  Function `FORM_HLRF`.
- Subset simulation (SuS) with adaptive conditional sampling (aCS) [1, 11]:
  Function `SuS`.
- Cross entropy-based importance sampling (IS) using different IS densities (single Gaussian, Gaussian mixture, von Mises-Fisher-Nakagami mixture) [6, 9]:
  Functions `CEIS_SG`, `CEIS_GM`, and `CEIS_vMFNM`.
- Sequential importance sampling (SIS) using different MCMC algorithms (independent M-H with Gaussian mixture proposal, adaptive CS) [10]:
  Function `SIS_GM`, `SIS_aCS`.

For further questions or bugs in the codes please write to:
`felipe.uribe@tum.de`.

## 2 Example 1

This problem is proposed in [4]. Due to its simplicity and flexibility it can be used to investigate the behavior of the reliability methods in varying dimensions. We consider a LSF expressed as a linear function of independent Gaussian random variables:

$$g(\boldsymbol{u}) = -\frac{1}{\sqrt{d}} \sum_{i=1}^{d} u_i + \beta \tag{2}$$

where, $\boldsymbol{u}$ is a $d$-dimensional standard Gaussian random vector. The probability of failure is independent of the dimension $d$ and can be derived analytically as $\mathrm{P}(\mathcal{F}) = \Phi(-\beta)$, where $\Phi(\cdot)$ is the standard Gaussian cumulative distribution function (CDF).

Figure 1 shows the LSF in two dimensions; the probability of failure in this case is $\mathrm{P}(\mathcal{F}) \approx 10^{-3}$ for $\beta = 3.0902$.

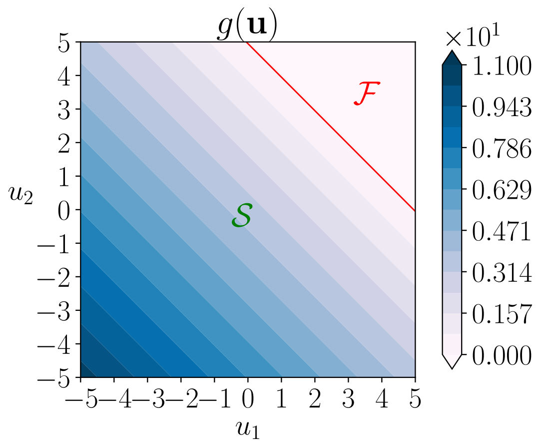


Figure 1: Example 1: limit state function in Eq. (2) with $d = 2$ and $\beta = 3.0902$.

## 3 Example 2

This problem is proposed in [7, 8]. We consider the LSF:

$$g(\boldsymbol{x}) = \pm \sum_{i=1}^{d} x_i \pm c \tag{3}$$

where, $\boldsymbol{x}$ is a $d$-dimensional vector of independent exponentially distributed random variables with parameter $\lambda = 1$. After transforming the vector $\boldsymbol{x}$ to standard Gaussian variables $\boldsymbol{u}$, the resulting LSF is highly nonlinear [5],

$$g(\boldsymbol{u}) = \pm \frac{1}{\lambda} \sum_{i=1}^{d} \ln(\Phi(-u_i)) \pm c$$

The probability of failure can be computed analytically as $\mathrm{P}(\mathcal{F}) = 1 - F_\Gamma(c; d, \lambda)$, where $c$ is a target threshold level and $F_\Gamma$ denotes the CDF of the gamma distribution with shape $d$ and scale $\lambda$. The negative sign in Eq. (3) corresponds to the lower tail of $F_\Gamma$ and the positive sign to the upper tail of $F_\Gamma$, respectively.

## 4 Example 3

This problem consists of a series system involving a linear and a convex LSFs in a two-dimensional standard Gaussian space [6]. This example is used for investigating the performance structural reliability methods in system problems with limit state functions of different type. The LSF is given by,

$$g(\boldsymbol{u}) = \min \begin{cases} 3.2 + \dfrac{(u_1+u_2)}{\sqrt{2}} \\ 0.1(u_1-u_2)^2 - \dfrac{(u_1+u_2)}{\sqrt{2}} + 2.5 \end{cases} \tag{4}$$

Figure 2 shows the LSF; the reference probability of failure in this case is $\mathrm{P}(\mathcal{F}) = 4.90 \times 10^{-3}$.

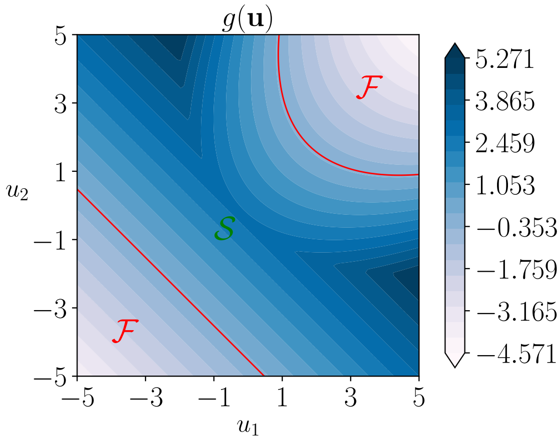


Figure 2: Example 1: limit state function in Eq. (4).

# 5 Example 4

This example is proposed in [3]. The LSF has a parabolic shape concave to the origin of the two-dimensional standard Gaussian space and has two design points; it is given by

$$g(\boldsymbol{u}) = 5 - u_2 - 0.5(u_1 - 0.1)^2 \tag{5}$$

Figure 3 shows the LSF; the reference probability of failure is $\mathrm{P}(\mathcal{F}) = 3.01 \times 10^{-3}$.

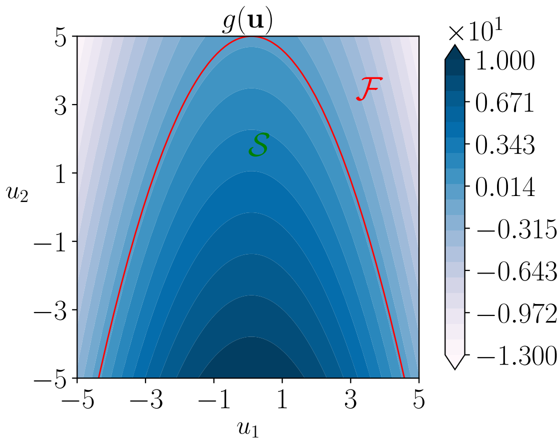


Figure 3: Example 1: limit state function in Eq. (5).

## 6 Example 5

This example is originally presented in [14]. We define a series system with four branches such that the LSF in the two-dimensional Gaussian space is expressed as

$$g(\boldsymbol{u}) = \min \begin{cases} 0.1(u_1 - u_2)^2 - \dfrac{(u_1 + u_2)}{\sqrt{2}} + 3 \\ 0.1(u_1 - u_2)^2 + \dfrac{(u_1 + u_2)}{\sqrt{2}} + 3 \\ u_1 - u_2 + \dfrac{7}{\sqrt{2}} \\ u_2 - u_1 + \dfrac{7}{\sqrt{2}} \end{cases} \tag{6}$$

Two of the components feature a parabolic limit state surface convex to the origin, while the remaining two components are of the linear type.

We depict the LSF in Figure 4. The reference probability of failure is estimated with Monte Carlo simulation ($N = 1 \times 10^6$) as $\mathrm{P}(\mathcal{F}) = 2.26 \times 10^{-3}$.

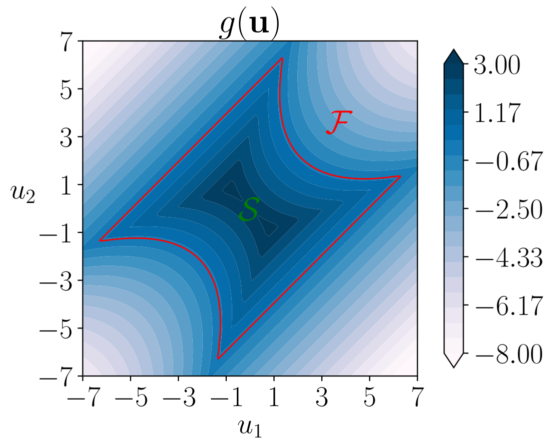


Figure 4: Example 5: limit state function in Eq. (6).

## 7 Example 6

This example is presented in [2]. Let $\boldsymbol{u}$ be a $d$-dimensional vector of independent standard Gaussian random variables. The failure domain is defined as all points $\boldsymbol{u}$ located outside of a hypersphere that has radius $r \in \mathbb{R}_{>0}$,

$$g(\boldsymbol{u}) = 1 - \frac{\|\boldsymbol{u}\|^2}{r^2} - \frac{u_1}{r}\left[\frac{1 - \left(\frac{\|\boldsymbol{u}\|}{r}\right)^m}{1 + \left(\frac{\|\boldsymbol{u}\|}{r}\right)^m}\right],$$

for $m \in [0, 4]$ the failure region is independent of $m$. The coefficient $m$ modifies the gradient of the limit state function in $u_1$-direction.

The sum of squared independent standard Gaussian random variables follows a chi-squared distribution. Therefore, the analytical solution of the probability of failure is given by [2],

$$\mathrm{P}(\mathcal{F}) = 1 - \frac{\gamma(d/2, r^2/d)}{\Gamma(d/2)} = \frac{\Gamma(d/2, r^2/d)}{\Gamma(d/2)} \tag{7}$$

where, $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function, and $\gamma(\cdot,\cdot)$ is the upper incomplete gamma function.

We depict the LSF in Figure 5 for dimension $d = 2$, radius $r = 5.26$, and $m = 2$. The reference probability of failure is $\mathrm{P}(\mathcal{F}) = 1 \times 10^{-6}$.

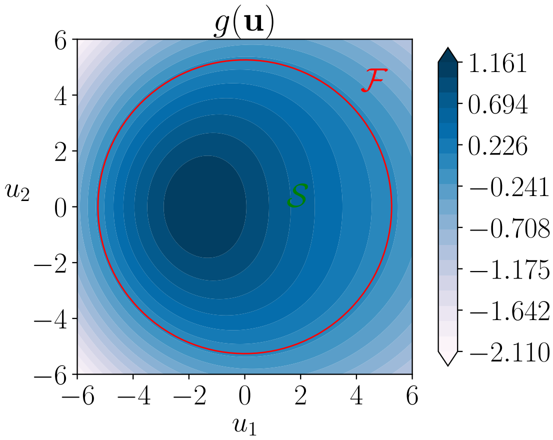


Figure 5: Example 6: limit state function in Eq. (7).

## References


[1] Au, S.-K., and J. L. Beck (2001). "Estimation of small failure probabilities in high dimensions by subset simulation". In: *Probabilistic Engineering Mechanics* 16.4, pp. 263-277.

[2] Betz, W. (2018). "Bayesian inference of engineering models". Ph.D. Thesis, Ingenieurfakultät Bau Geo Umwelt, Technische Universität München.

[3] Der Kiureghian, A., and T. Dakessian (1998). "Multiple design points in first and second-order reliability", Structural Safety 20.1, pp. 37-49.

[4] Engelund, S., and R. Rackwitz (1993). "A benchmark study on importance sampling techniques in structural reliability". In: *Structural Safety* 12.4, pp. 255-276.

[5] Fujita, M., and R. Rackwitz (1988). "Updating first- and second-order estimates by importance sampling". *Structural Engineering/Earthquake Engineering* 5.1, pp. 53-59.

[6] Geyer, S., I. Papaioannou, and D. Straub (2018). "Cross entropy-based importance sampling using Gaussian densities revisited". Structural Safety (in review).

[7] Hohenbichler, M., and R. Rackwitz (1983). "Non-normal dependent vectors in structural safety". *Journal of the Engineering Mechanics Division* 107.6, pp. 1227-1238.

[8] Hohenbichler, M., and R. Rackwitz (1988). "Improvement of second-order reliability estimates by importance sampling". *Journal of Engineering Mechanics* 114.12, pp. 53-59.

[9] Papaioannou, I., S. Geyer, and D. Straub. "A new flexible mixture model for cross entropy based importance sampling". In preparation.

[10] Papaioannou, I., C. Papadimitriou, and D. Straub (2015). "Sequential importance sampling for structural reliability analysis". In: *Structural Safety* 62, pp. 66-75.

[11] Papaioannou, I., W. Betz, K. Zwirglmaier, and D. Straub (2015). "MCMC algorithms for Subset Simulation". In: *Probabilistic Engineering Mechanics* 41, pp. 89-103.

[12] Rackwitz, R. (1976). Practical probabilistic approach to design. First order reliability concepts for design codes. Bulletin d'Information du CEB, Vol. 112.

[13] Rackwitz, R., and B. Fiessler (1979). Structural reliability under combined random load sequences. Computers and Structures, 9.5, pp 489-494.

[14] Waarts, P.-H. (2000). "Structural reliability using finite element methods. An appraisal for DARS: Directional Adaptive Response surface Sampling". Ph.D. thesis. Technical University of Delft.