# HIGH-ORDER ONE-STEP SCHEMES FOR NON-CONSERVATIVE PDE: APPLICATION TO SHALLOW WATER SYSTEMS

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## Abstract

We show applications of a new unified family of arbitrary high order accurate *path-conservative* one--step schemes on unstructured triangular and adaptively refined Cartesian meshes for the solution of hyperbolic PDE with non-conservative products and stiff source terms. The fully discrete one-step schemes are within the general framework of $P_NP_M$ schemes first proposed in [6].

In this general framework, classical high order finite volume and discontinuous Galerkin schemes are only special cases. The one-step time discretization of high order of accuracy is obtained using a new local space-time Galerkin predictor that is also able to deal with stiff source terms [5]. The centered treatment of the non-conservative products is done using a multi-dimensional generalization of the FORCE scheme, see [2;6]. We show applications of our new method to shallow water equations.

## Introduction

Most state of the art of free surface flow models commonly used in environmental engineering and geophysics are based on some kind of depth–averaged shallow water type flow model. The most basic two–dimensional shallow water model with fixed bed, without friction and with only one single layer of liquid is given by the following nonlinear PDE system:

$$\frac{\partial \boldsymbol{W}}{\partial t} + \boldsymbol{A}(\boldsymbol{W})\nabla \boldsymbol{W} = \boldsymbol{S}(W), \tag{1}$$

where $\boldsymbol{W}$ is the state vector, $\boldsymbol{A}(\boldsymbol{W})$ is system matrix and $\boldsymbol{S(W)}$ is the source term, which may also be stiff. In the special case where $\boldsymbol{A}(\mathbf{W})$ is the Jacobian of a flux $\boldsymbol{F(W)}$, then equation (1) reduces to a classical conservation law

$$\frac{\partial \boldsymbol{W}}{\partial t} + \nabla \boldsymbol{F}(\boldsymbol{W}) = \boldsymbol{S}(\boldsymbol{W}) \tag{2}$$

Up to now, the principle of conservation is the most powerful tool available to give insight into physical processes observed in nature. However, many physical processes, in particular multi-fluid or multi-phase flows, cannot be written as conservation laws, but important non--conservative terms that model the phase interactions remain. So, one is obliged to use the more general PDE (1).

In this presentation we follow a *centered* philosophy, i.e. our schemes will not use any upwinding and hence will not need any wave--propagation information. Such schemes have been recently published in [1;2;3;6;10], where all the details can be found. Here, we only give the final formulation of the scheme and show some applications to standard test cases.

## FORCE scheme

In the present paper the computational domain $\Omega$ is discretized by, non--overlapping elements $T_i$. In the case of adaptive mesh-refinement, there may also be overlapping nodes. We introduce the following operators,

$$\langle f,g\rangle_{T_i} = \int_{t^n}^{t^{n+1}} \int_{T_i} f(\vec{x},t)g(\vec{x},t)dVdt$$

$$[f,g]_{T_i}^{t} = \int_{T_i} f(\vec{x},t)g(\vec{x},t)dV$$

$$\{f,g\}_{T_i} = \int_{t^n}^{t^{n+1}} \int_{\partial T_i} f(\vec{x},t)g(\vec{x},t)dSdt \tag{3}$$

which denote the scalar products of two functions $f$ and $g$ over the space-time element $T_i \ x \ [t^n;\ t^{n+1}]$, the spatial element $T_i$, and the space-time boundary element $\partial T_i \times [t^n;t^{n+1\backslash}]$ respectively.

Taking PDE (1) and multiplying it with a space-only dependent test function $\Phi_k$, we obtain the following one-step $P_NP_M$ scheme:

$$[\Phi_k, \mathbf{u_h}]_{T_i}^{t^{n+1}} - [\Phi_k, \mathbf{u_h}]_{T_i}^{t^n} + \langle \Phi_k, \boldsymbol{A}(\boldsymbol{\mathcal{W}_h}).\nabla \boldsymbol{\mathcal{W}_h}\rangle_{T_i \backslash \partial T_i} + \{\boldsymbol{\Phi}_k, \boldsymbol{\mathcal{D}}^{-}(\boldsymbol{\mathcal{W}}_h^{-}, \boldsymbol{\mathcal{W}}_h^{+}, \mathbf{n})\}_{\partial T_i} = \langle \boldsymbol{\Phi}_k, \boldsymbol{S}(\boldsymbol{\mathcal{W}_h})\rangle_{T_i} \tag{4}$$

Here, $\boldsymbol{u_h}$ are piecewise polynomials of degree $N$, $\boldsymbol{\mathcal{W}_h}$ is the numerical solution of the space-time Galerkin predictor method introduced in [4,5,6], $\boldsymbol{\mathcal{W}}_h^{-}, \boldsymbol{\mathcal{W}}_h^{+}$ are the boundary extrapolated values of the element $T_i$ and its neighbor, respectively, and $\boldsymbol{\mathcal{D}}^{-}(\boldsymbol{\mathcal{W}}_h^{-}, \boldsymbol{\mathcal{W}}_h^{+}, \mathbf{n})$ denotes the jump term, which for our central schemes is simply defined as

$$\boldsymbol{\mathcal{D}}^{-} = \frac{1}{2}\left(\tilde{\mathbf{A}}_{\boldsymbol{\Psi}}^{\mathbf{G}} - \beta_j^{LW}\left(\tilde{\mathbf{A}}_{\boldsymbol{\Psi}}^{\mathbf{G}}\right)^2 - \beta_j^{LF}\mathbf{I}\right)(\boldsymbol{\mathcal{W}}_h^{+} - \boldsymbol{\mathcal{W}}_h^{-}) \tag{5}$$

where $\mathbf{I}$ is the identity matrix and the two constants $\boldsymbol{\beta_j^{LW}}$ **and** $\boldsymbol{\beta_j^{LF}}$ contain only information on the geometry and the time step $\Delta t$, as derived in detail in [11;7]:

$$\beta_j^{LF} = \frac{2}{\Delta t S_j}\frac{V_j^{-}V_j^{+}}{V_j^{-} + V_j^{+}}, \qquad \beta_j^{LW} = \frac{1}{2}\frac{\Delta t S_j}{V_j^{-} + V_j^{+}}. \tag{6}$$

Here, $\boldsymbol{V_j^{\pm}}$ denote the sub-volumes defined at each edge/face j, where $\boldsymbol{V_j^{-}}$ is the sub-volume inside the considered element and $\boldsymbol{V_j^{+}}$ is the corresponding sub-volume in the neighbor adjacent to edge $j$. Throughout this paper, we define the path $\boldsymbol{\Psi}$ by the family of segments

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}(\boldsymbol{\mathcal{W}}_h^{-}, \boldsymbol{\mathcal{W}}_h^{+}, s) = \boldsymbol{\mathcal{W}}_h^{-} + s(\boldsymbol{\mathcal{W}}_h^{+} - \boldsymbol{\mathcal{W}}_h^{-}) \tag{7}$$

The matrix $\tilde{\mathbf{A}}_{\boldsymbol{\Psi}}^{\mathbf{G}}$ is defined by the following path integral, which is computed numerically using an appropriate high order Gaussian quadrature rule:

$$\tilde{\mathbf{A}}_{\boldsymbol{\Psi}}^{\mathbf{G}} = \int_0^1 \boldsymbol{A}\left(\boldsymbol{\Psi}(\boldsymbol{\mathcal{W}}_h^{-}, \boldsymbol{\mathcal{W}}_h^{+}, s)\right)ds \tag{8}$$

## Osher-type schemes

Instead of using the FORCE scheme presented above, another option is to use the extension of the well-known Riemann solver of Osher, [8;11], to a certain class of hyperbolic systems in non-conservative form, in particular to shallow-water-type and multi--phase flow models, as developed in [8]. In the case of Osher-type schemes, the jump terms $\boldsymbol{\mathcal{D}}_{i+1/2}^{\pm}$ are defined as follows

$$\boldsymbol{\mathcal{D}}_{i+1/2}^{\pm} = \frac{1}{2}\int_0^1 \boldsymbol{A}\left(\boldsymbol{\Psi}(\boldsymbol{W}_{i+1/2}^{-}, \boldsymbol{W}_{i+1/2}^{+}, s)\right) \pm \left|\boldsymbol{A}\left(\boldsymbol{\Psi}(\boldsymbol{W}_{i+1/2}^{-}, \boldsymbol{W}_{i+1/2}^{+}, s)\right)\right|\frac{d\Psi}{ds}ds \tag{9}$$

The first term in equation (9) is the centered part of the numerical flux and the second term is the numerical viscosity, which is given in terms of a path-integral. Here, the path $\boldsymbol{\Psi}(\boldsymbol{W}_{i+1/2}^{-}, \boldsymbol{W}_{i+1/2}^{+}, \boldsymbol{s})$ with $\boldsymbol{0 \le s \le 1}$ that connects the left state $\boldsymbol{W}_{i+1/2}^{-}$ with the right state $\boldsymbol{W}_{i+1/2}^{+}$, in phase-space is a Lipschitz continuous function.

Following the same choice of paths as in (7) the jump term (9) will read

$$\boldsymbol{\mathcal{D}}_{i+1/2}^{\pm} = \frac{1}{2}\left(\int_0^1 (\boldsymbol{A}(\boldsymbol{\Psi}) \pm |\boldsymbol{A}(\boldsymbol{\Psi})|)ds\right)\left(\boldsymbol{W}_{i+1/2}^{+} - \boldsymbol{W}_{i+1/2}^{-}\right) \tag{10}$$

The integrals appearing in (10) may again be evaluated by means of any numerical quadrature formula.

## Adaptative mesh refinement

A useful technique when coding the numerical scheme is to refine locally wherever it is needed. The technique used in this work is the so-called Adaptive Mesh Refinement (AMR). It allows to automatically refine the mesh during the calculation progress, when a more accurate resolution in certain parts of the domain is needed. The algorithm developed here is dynamic, in the sense that the refinement process tracks the solution, refining and de-refining when necessary. This process saves CPU time and also allows to obtain a more accurate solution compared to a uniform mesh approach.

## Computational Results

### Well-balanced property

The first test problem is used to verify that the proposed numerical algorithm is well-balanced. The well-balanced or so-called C-property requires that water at rest should remain at rest also on the discrete level, independent of the particular bottom geometry. Here, we solve a two dimensional problem that consists of a small perturbation of the free surface of a lake at rest. The bottom of the lake contains a bump. The results are depicted in Figure 1 and correspond well with the reference solutions found in literature [1]. Without perturbation, the steady state is maintained up to machine precision, see Table 1, where the error norms $L_1$ and $L_\infty$ for both the water height (H) and the water flow rate (q) are depicted.

| Testcase | $H$ (m) | | $q$ (m$^2$/s) | |
|---|---|---|---|---|
| | $L_1$ | $L_\infty$ | $L_1$ | $L_\infty$ |
| Test 1 (smooth) | 2.31E−15 | 1.23E−14 | 2.12E−15 | 5.46E−14 |
| Test 2 (non-smooth) | 5.35E−15 | 3.29E−14 | 6.44E−15 | 2.33E−14 |

Table 1: Discrete well-balanced property for smooth and non-smooth irregular bottom geometry.

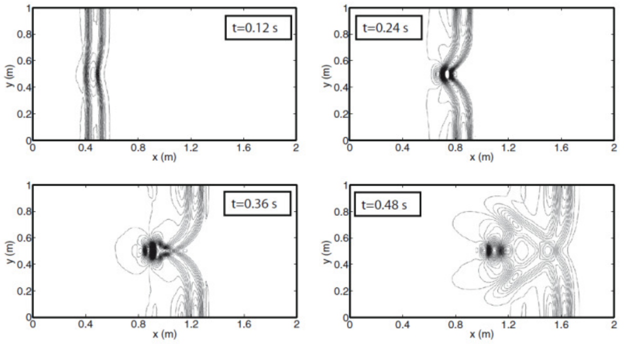


Figure 1: Small perturbation of a two-dimensional steady state water travelling over a bottom bump.

### Internal Dam-Break for a Two-Layer Shallow Water System

The next test problem consists in an internal dam-break problem for the two-layer shallow water equations, see [12] for details, over a flat and variable bottom. There is no exact solution available, but we can compare with the results of the path-conservative Roe-type schemes proposed by Castro and Parés in [12]. As we can deduce from the results depicted in Figures 2 and 3 we obtain an excellent agreement. The numerical scheme used is based on the ADER (Arbitrary order DErivative Riemann problem) approach [11; 5].

### Circular Dam Break on Adaptively Refined Mesh

In this last section we present some preliminary results of our new high order one-step path-conservative WENO finite volume schemes on adaptively refined Cartesian meshes. The method uses time-accurate *local* time stepping [13], in order to increase computational efficiency. This means that the smallest elements of the computational domain run on a small time step that is given by a local Courant stability criterion, while the large cells are allowed to run large time steps according to their local CFL condition. The use of our particular one-step time discretization based on a local space-time Galerkin predictor scheme allows us to compute the interface fluxes *without* needing any time-interpolation and without any complicated bookkeeping. The computational overhead needed for the AMR compared to a fixed mesh approach is therefore quite minimal. Here we show the AMR mesh (Figure 4) and the results (Figure 5) for a two-dimensional circular dam break problem.

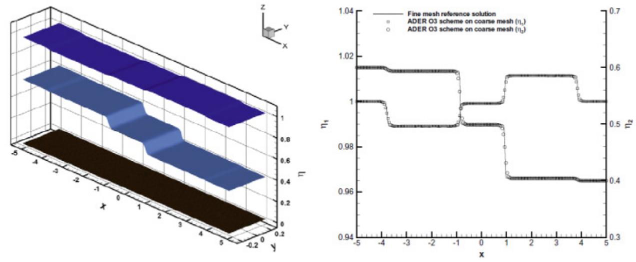

Figure 2: Two-layer shallow water dambreak problem over a flat bottom. 3D visualization of the results (left) and comparison with the reference solution of Castro and Parés [12] (right).

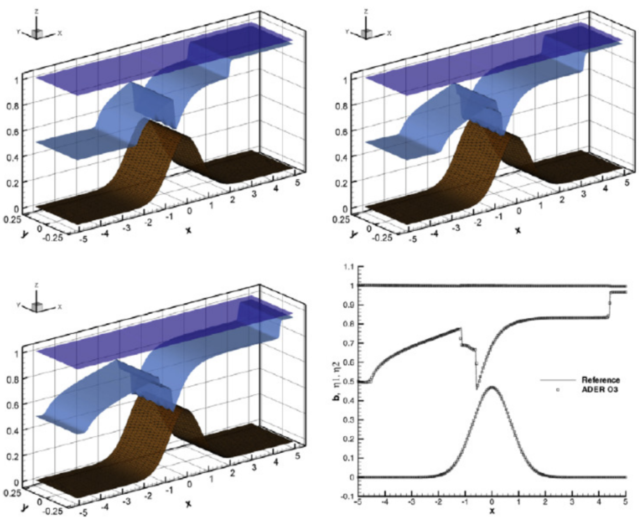

Figure 3: Two-layer shallow water dambreak problem over a smooth bottom bump. 3D visualization of the results (left) and comparison with the reference solution of Castro and Parés [12] (right).

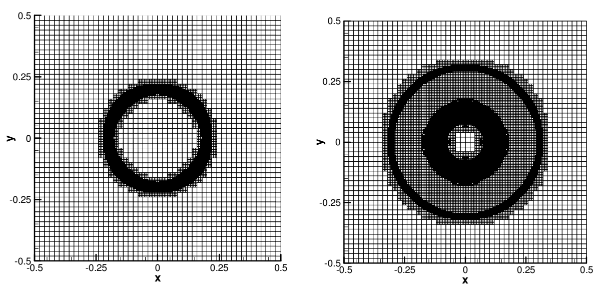

Figure 4: AMR mesh for a two-dimensional circular dam-break problem of the one-layer shallow water equations.

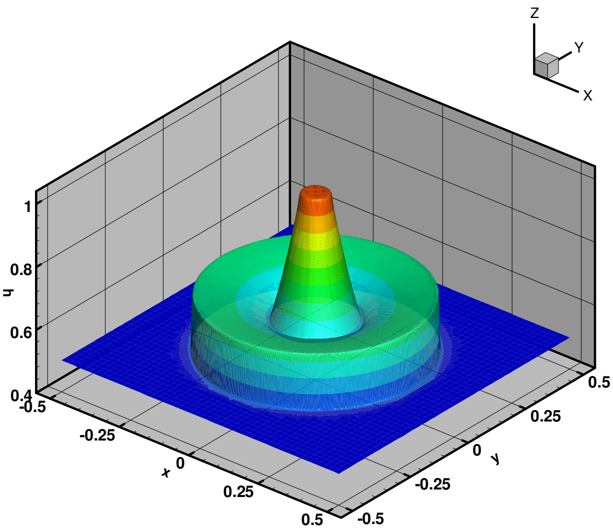


Figure 5: Three-dimensional visualization of the results obtained for the water depth in the two-dimensional circular dam-break problem.

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