STATISTICAL MOMENT APPROACH FOR THE STUDY OF NON-UNIFORM SEDIMENT TRANSPORT IN ONE DIMENSIONAL FREE SURFACE FLOWS

Giulia Garegnani & Giorgio Rosatti

Department of Civil and Environmental Engineering, University of Trento, Italy, Via Mesiano 77, 38123

E-mail: giulia.garegnani@ing.unitn.it

Abstract

In unsteady flows, the uniform sediment transport is commonly described by solving the conservation equations of mass and momentum both for solid and liquid phases. If the non-uniformity of the sediment becomes relevant, a new set of equations, regarding the time and space evolution of the grain size distribution of the solid phase, must be considered. Two approaches are presented in literature. The bed material fraction (BMF) model discretizes the grain size distribution curve in a finite number of classes while the statistical moment (SM) approach, proposed by Armanini, (1992) and (1995), describes the distribution curve by means of its moments (commonly means and variance). Herein a rigorous analytical derivation of the SM equations is proposed. The two models are implemented in the case of 1D rectangular channel with unitary width and compared. Advantages, limits of the two approaches and some preliminary numerical results are herein presented.

Introduction

Rivers-beds are usually composed of non-uniform sediment mixtures and the prediction of natural river processes is, consequently, more complex with respect to a uniform approach.

In order to model the dynamic of the non-uniform sediment transport, in the most used mathematical and numerical models, the concept of mixing layer (or active layer) is introduced. This concept was proposed by Hirano (1971) and it assumes that in this mixing layer all grains with different diameters are instantaneously and fully mixed. Indeed the bed river results subdivided in two layers, the mixing layer and the substrate, with two different grain size distributions. In order to study how these distributions evolve in time and space in function of the hydrodynamics and of the sediment, conservation equations for the solid phase in the mixing layer must be written.

In the bed material fraction (BMF) models the grain size distributions curve are divided in several classes and as many mass conservation equations computed (Wu, 2007) (Brunner, 2010) (Lee & Hsieh, 2003) (Yang & Simoes, 2002). Indeed the number of unknowns are equal to the number of classes in which the grain size distribution curve is divided.

In the model statistical moment (SM) approach, starting from the conservation equation of the solid phase in the mixing layer a set of equations for the statistical moments is derived (Armanini, 1992) and (Armanini, 1995). The advantage of this formulation is the reduction of the unknowns from the number of classes necessary to describe the grain-size distribution to the number of statistical moments with a consequent reduction of the computational cost.

Mathematical modeling of non-uniform sediment transport

In this work a 1D rectangular channel with unit width is considered. Mathematical modeling of river mobile bed is usually based on the vertically averaged Saint-Venant equations, which express the conservation laws of mass and momentum for water, and on the Exner equation resulting from the conservation of the solid phase. These equations can be also derived from a two phase approach under the assumption of low sediment concentration and isokinetic model (Garegnani, Rosatti, & Bonaventura, 2011).

The momentum equation thus reduces to the equation for the clear water:

$$\frac{dq}{dt}+\frac{d}{dx}\left(\frac{q^2}{h}\right)+gh\frac{d\eta}{dx}=-\frac{\bar{\tau}_w}{\rho_w} \qquad (1)$$

where q is the mixture discharge that coincides with the liquid discharge while x and t are the longitudinal coordinate and the time variable respectively. Herein g is the gravity acceleration, h the water depth, ρ_w the water density and $\bar{\tau}_w$ the average bottom friction. In equation (1) the friction term is calculated through the Gauckler-Strickler relation:

$$\frac{\bar{\tau}_w}{\rho} = \frac{g}{k_s^2 h^{4/3}} q \tag{2}$$

where k_s is the Strickler roughness coefficient. Besides, the momentum equation (1) can be rewritten in a quasi-lagrangian formulation:

$$\frac{Dq}{Dt} + gh\frac{d\eta}{dx} = -\alpha q \tag{3}$$

where using equation (2) yields

$$\alpha = \frac{g}{k_s^2 h^{4/3}} + \frac{\partial u}{\partial x} \tag{4}$$

with $u = q/h$ velocity of the mixture.
The conservation of the total mass is:

$$\frac{d\eta}{dt} + \frac{dq}{dx} = 0 \tag{5}$$

where η is the free surface elevation (see Figure 1) .

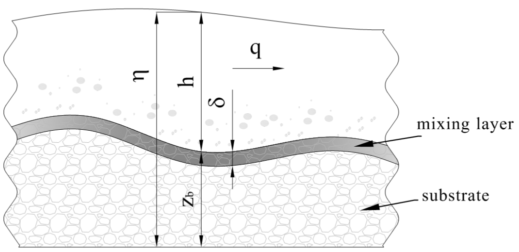

Figure 1: Sketch of the main model variable in the case of non-uniform sediment.

Finally, under the hypothesis of low sediment concentration, the time variation of the sediment stored in the water column is neglected and the conservation equation for the sold phase is:

$$c_b\frac{dz_b}{dt} + \frac{d}{dx}(cq) = 0. \tag{6}$$

Here c_b is the concentration of the sediment in the bed and it is assumed to be constant while cq is the solid discharge. Notice that c is indeed defined as the ratio of solid discharge to the total discharge, q_s/q. Indeed, it is not the volumetric concentration of sediment in the column water that can be calculated from c by means of a corrective coefficient taking into account the vertical distribution of the velocity and of the concentration.
The free surface elevation η, the mixture discharge q and the bed level z_b are the unknowns of the system of equations (3), (5) and (6). In the equation (6) the sediment concentration c is, in the case of uniform transport, calculated through a sediment transport formula as shown in (Garegnani, Rosatti, & Bonaventura, 2011) while, in the case of non-uniformity of the sediment, c depends on the grain size distribution as shown in the following section. In this case, a new set of equation has to be added to the system (3), (5) and (6) in order to calculate the concentration c and to study the evolution of the grain-size distribution curve.

The grain-size distribution curve

Sieve analyses allow to estimate the distribution of granular material on the bed. In Figure 2 the typically lognormal distribution of the grains is reported. Grain-size distribution is a continuous concept. The sample space is the space of the diameters ϕ. The grain-size distribution function $F(\phi)$ gives the probability that a random diameter is not larger than a given value ϕ.

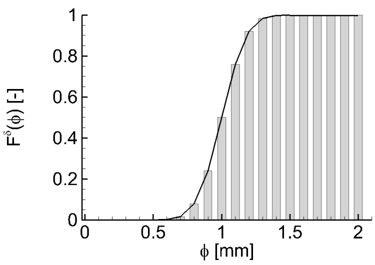

Figure 2: Grain-size distribution function $F^\delta(\phi)$ in the mixing layer δ.

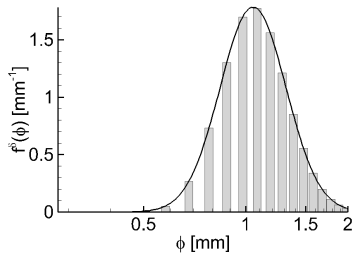

Figure 3: Density function $f^\delta(\phi)$ in the micing layer.

Indeed, we introduce the concept of density function $f(\phi) = dF/d\phi$ where $f(\phi)d\phi$ is the probability that a random value of diameter lies between ϕ and $\phi + d\phi$, Figure 3.
In order to study the dynamic of the non-uniform transport, the bed river is subdivided in two layers: the mixing layer with thickness δ and the substrate with depth $z_b - \delta$ as shown in Figure 1. Generally the grain size density function $f^\delta(\phi)$ in the mixing layer differs from the density curve $f^{subst}(\phi)$ in the substrate while the concentration in the two layers is considered constant and equal to c_b. Besides,

the density curve $f^t(\phi)$ of the material transported by the flow changes from either the mixing layer and the substrate. If the sediment distribution in the mixing layer is known, the Einstein assumption allow to calculate the sediment concentration. This hypothesis affirms that interactions among the moving sediment particles are negligible. Indeed, the density of the solid concentration c depends on the grain-size distribution present in the mixing layer:

$$c(\phi) = f^\delta(\phi) c_c(\phi) \tag{7}$$

where c_c is the solid concentration evaluated in the case of uniform flow and grain size material equal to ϕ, i.e. through a sediment transport formula. In this work a monomial formula is used:

$$c_c(\phi) = m \frac{u^{k-1}}{\phi^l}. \tag{8}$$

Notice that the density function $f^t(\phi)$ of the material transported by the flow differs from the curve $f^\delta(\phi)$ in the mixing layer and it is equal to:

$$f^t(\phi) = \frac{f^\delta(\phi) c_c(\phi)}{c} \tag{9}$$

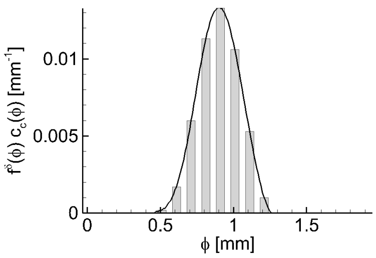

Figure 4: Concentration density $c(\phi)$ of the material transported by the flow. The total solid concentration is calculated as the area of the region bounded by the graph.

The total solid concentration c is the integral over the grain-size interval of equation (7), Figure 4:

$$c = \int_0^{+\infty} f^\delta(\phi) c_c(\phi) d\phi. \tag{10}$$

Mass conservation in the mixing layer

The conservation equation for sediments with diameter ϕ in the mixing layer δ is then

$$\frac{\partial}{\partial t}\left(f^\delta c_b \delta\right) + \frac{\partial}{\partial x}\left(f^\delta c_c q\right) + f^* c_b \frac{\partial}{\partial t}(z_b - \delta) = 0 \tag{11}$$

where the first term is the time variation of the volume occupied by the sediment with diameter ϕ, the second term is the spatial variation of the solid discharge and the last is the flux of sediment from the substrate to the mixing layer. The flux between mixing layer and substrate depends on the grain size density function f^δ in the mixing layer in the case of bed aggradations and on f^{subst} of the substrate in the case of bed degradations.

The thickness δ of the mixing layer is calculated through a closure formula, it is usually related to the sand dunes height as (Armanini, 1999) and (Wu, 2007) or to the specific diameter ϕ_{90} (Armanini & Di Silvio, 1988) as:

$$\delta \approx (2 \div 3)\phi_{90}. \tag{12}$$

The substrate is assumed to be unbounded and, then, the grain size distribution curve of the sediment inside it doesn't change and $f^{subst}(\phi)$ is constant in time.

The BFM model

The space of diameters ϕ is divided in m intervals $j = [\phi_{j-1/2}, \phi_{j+1/2}]$. If the mixing layer is considered, in the discrete case, the probability that ϕ is included within the interval $j = [\phi_{j-1/2}, \phi_{j+1/2}]$. is:

$$p_j = \int_{\phi_{j+1/2}}^{\phi_{j-1/2}} f^\delta(\phi') d\phi'. \tag{13}$$

Equation (11) is integrated on the j-interval $[\phi_{j-1/2}, \phi_{j+1/2}]$:

$$\int_{\phi_{j-1/2}}^{\phi_{j+1/2}} \left[\frac{\partial}{\partial t}\left(f^\delta c_b \delta\right) + \frac{\partial\left(f^\delta c_c q\right)}{\partial x} + f^* c_b \frac{\partial}{\partial t}(z_b - \delta)\right] d\phi = 0. \tag{14}$$

From equation (13), we obtain the equation of mass conservation for the j-class on the interval j:

$$c_b \frac{\partial}{\partial t}\left(p_j \delta\right) + \frac{\partial}{\partial x}\left[p_j c_c(\phi_j) q\right] + c_b p_j^* \frac{\partial}{\partial t}(z_b - \delta) = 0 \tag{15}$$

with $j = 1 \dots m$ where the unknowns are the probability p_j in the mixing layer. Notice that p_j is also called availability factor because it represents the fraction of materials available in the mixing layer. The availability factors p_j^* are equal to the value p_j in the mixing layer in the case of bed aggradations and to the value of the availability factor p_j^{subst} of the substrate in the case of bed degradations.

The system of m equations (15) can be numerically solved in order to model the river processes.

The solid discharge in equation (6) is the sum of all class contributions and it is calculated by the discretization of equation (10):

$$c = \sum_{j=1}^{m} p_j (c_c)_j. \tag{16}$$

An analytical derivation of the SM model

This novel approach was proposed by Armanini (1992) and (1995). The objective of this formulation is the reduction of the number of unknowns from the number m of classes in which the distribution curve is discretized to the number of statistical moments necessary to describe the shape of the distribution curve. The average $\mu(x,t)$, the variance $\sigma^2(x,t)$ and all the statistical moments considered can change in time and space and, consequently, also the shape of the density function f.

In this work we consider only changes in the mean value but the following procedure for deriving the moment equation for the mean is suitable for higher moments.

The definition of mean value and variance respectively are:

$$\begin{aligned} &first\ moment = \mu = \int_{\phi=0}^{\phi=+\infty} \phi d\phi; \\ &second\ central\ moment = \sigma^2 = \\ &\qquad = \int_{\phi=0}^{\phi=+\infty} (\phi-\mu)^2 d\phi. \end{aligned} \tag{17}$$

Similarly to the BFM model, the conservation equation for the sediment with diameter ϕ, equation (11), is multiplied by ϕ and integrated over all the sample space:

$$\begin{aligned} \int_0^{+\infty} \phi \left[\frac{\partial}{\partial t}\left(f^\delta \mathrm{c}_b \delta\right) + \frac{\partial\left(f^\delta c_c q\right)}{\partial x} \right. \\ \left. + f^* \mathrm{c}_b \frac{\partial}{\partial t}(z_b - \delta) \right] d\phi = 0. \end{aligned} \tag{18}$$

By remembering that the diameter ϕ is an independent variable, the first and the last term of the equation (18) are:

$$\begin{aligned} &\int_0^{+\infty} \frac{\partial}{\partial t}\left(f^\delta \phi c_b \delta\right) d\phi = c_b \frac{\partial}{\partial t}(\mu\delta), \\ &\int_0^{+\infty} \phi f^* c_b \frac{\partial}{\partial t}(z_b - \delta)\, d\phi = \mu^* c_b \frac{\partial}{\partial t}(z_b - \delta) \end{aligned} \tag{19}$$

where μ^* is equal to the mean diameter of the substrate in the case of bed degradations and to μ in the case of bed aggradations. The second term of the equation (18) represents the flux of sediment that determines the variation of the mean value:

$$\int_0^{+\infty} \frac{\partial}{\partial x}\left(f^\delta \phi c_c q\right) d\phi = \frac{\partial}{\partial x}\left(F_\mu q\right). \tag{20}$$

The flux $F_\mu = \int_0^{+\infty}\left(f^\delta \phi c_c\right) d\phi$ can be expressed in function of the statistical moments by expending ϕc_c in Taylor series around the mean diameter:

$$\begin{aligned} F_\mu = \mu c_c(\mu) + \left(\left.\frac{\partial c_c}{\partial \phi}\right|_{\phi=\mu} + \frac{\mu}{2} \left.\frac{\partial^2 c_c}{\partial \phi^2}\right|_{\phi=\mu} \right) \sigma^2 \\ + o(\phi^3). \end{aligned} \tag{21}$$

The equation (18) becomes:

$$\mathrm{c}_b \frac{\partial}{\partial t}(\mu\delta) + \frac{\partial}{\partial x}\left(F_\mu q\right) + c_b \mu^* \frac{\partial}{\partial t}(z_b - \delta) = 0. \tag{22}$$

Finally, in the conservation equation for the solid phase, equation (6), the sediment concentration c should be expressed in function of the statistical moments. In order to calculate the integral (10), the capacity concentration $c_c(\phi)$ is expanded in Taylor series around the mean value μ until second-order partial derivatives:

$$\begin{aligned} c_c(\phi) = c_c(\mu) + \left.\frac{\partial c_c}{\partial \phi}\right|_{\phi=\mu} (\phi-\mu) \\ + \left.\frac{\partial^2 c_c}{\partial \phi^2}\right|_{\phi=\mu} (\phi-\mu)^2 + o(\phi^3). \end{aligned} \tag{23}$$

The capacity concentration is substituted into equation (10) to calculate the total concentration c and remembering that $\int_0^{+\infty} f(\phi) d\phi = 1$ yields:

$$c = c_c(\mu) + \frac{1}{2} \left.\frac{\partial^2 c_c}{\partial \phi^2}\right|_{\phi=\mu} \sigma^2 + o(\phi^3). \tag{24}$$

The procedure can be extend to all the statistical moments in order to better describe the time evolution of the grain-size distribution curve. In this preliminary work the variance σ^2 is considered constant and only changes in the mean value are studied.

The numerical scheme

In the case of BFM approach the resulting system is composed by equations (3), (5), (6) and (15) where the sediment concentration c in equation (6) is calculated through equation (16). The unknowns of the system are the free surface elevation η, the mixture discharge q, the bed level z_b and the m availability factors p_j. Instead, the SM approach includes equation (3), (5), (6) and (22). The concentration c in equation (6) is calculated by means of equation (24). The unknowns of the system are still the same of the BFM approach except for the availability

factors p_j that are substituted by the mean μ of the grain size distribution.
The computational domain is discretized by a staggered computational grid where the bed level z_b and the free surface elevation η are defined at the integer nodes x_i with $i = 1 \dots N$ while the discharge q is defined at the half integer nodes $x_{1+1/2} = \left(x_i + x_{i+1/2}\right)/2$. The values of water depth at the nodes $i + 1/2$ are computed by an upwind interpolation. The node distribution is arbitrary and the node spacing is defined as $\Delta x_i = x_{i+1/2} - x_{i-1/2}$.
The solution procedures for both the systems are similar. Equations (3), (5) and (6) are discretized following the procedure proposed in (Garegnani, Rosatti, & Bonaventura, 2011).
The discretization of equation (3) is substituted in equation (5) and we obtain a system where the only unknowns are the values of the free surface elevation η_i^{n+1} at the integer nodes *i* and at the time step $n + 1$. The system obtained is linear and can easily solved with a direct method. The calculated values of free surface are used in equation (3) to obtain the discharge $q_{i+1/2}^{n+1}$.
The continuity equation (6) for the solid mass is integrated in both the cases over $\left[x_{i+1/2}, x_{i-1/2}\right]$. The fluxes are discretized in time in a semi-implicit fashion while the concentration *c* is explicit:

$$\begin{aligned} c_b \Delta z_b^{n+1} &= c_b\left(z_b^{n+1} - z_b^n\right) \\ &= \frac{\vartheta \Delta t}{\Delta x}\left(c_{i-1/2}^n q_{i-1/2}^{n+1} - c_{i+1/2}^n q_{i+1/2}^{n+1}\right) \\ &+ \frac{(1-\vartheta)\Delta t}{\Delta x}\left(c_{i-1/2}^{n-1} q_{i-1/2}^n - c_{i+1/2}^{n-1} q_{i+1/2}^n\right) \end{aligned} \tag{25}$$

where $c_{i+1/2}^n$ is calculated through equation (16) for the BFM method and equation (22) for the SM method as reported in the following sections.

The BFM scheme

The availability factors are defined at the integer nodes *i*.
Under the assumption of δ constant in time, the equation (15) is integrated over the control volume $\left[x_{i+1/2}, x_{i-1/2}\right]$:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left[c_b \delta \frac{\partial p_j}{\partial t} + \frac{\partial}{\partial x}\left(p_j c_c(\phi_j) q\right) + c_b p_j^* \frac{\partial z_b}{\partial t}\right] dx = 0. \tag{26}$$

The first term of the previous equation is discretized by a forward-in-time finite difference method while, the second term, the gradient of solid discharge, is discretized in space by a centered finite difference and in time by a semi-implicit time-averaging:

$$p_{i,j}^{n+1} = p_{i,j}^n - \frac{\vartheta \Delta t}{c_b \delta \Delta x}\left[p_{i+\frac{1}{2},j}^n (c_c)_{i+\frac{1}{2},j}^n q_{i+\frac{1}{2}}^{n+1} - p_{i-\frac{1}{2},j}^n (c_c)_{i-\frac{1}{2},j}^n q_{i-\frac{1}{2}}^{n+1}\right] - \frac{(1-\vartheta)\Delta t}{c_b \delta \Delta x}\left[p_{i+\frac{1}{2},j}^{n-1} (c_c)_{i+\frac{1}{2},j}^{n-1} q_{i+\frac{1}{2}}^n - p_{i-\frac{1}{2},j}^{n-1} (c_c)_{i-\frac{1}{2},j}^{n-1} q_{i-\frac{1}{2}}^n\right] + (p^*)_{i,j}^n \frac{\Delta z_b^{n+1}}{\delta \Delta x} = 0. \tag{27}$$

The values of the availability factors at the nodes $i + 1/2$ are computed by an upwind interpolation while $(c_c)_{i+1/2,j}^n$ is calculated through equation (8). After evaluating the availability factors $p_{i+1/2,j}^{n+1}$, the concentration $c_{i+1/2}^{n+1}$ is computed by means of equation (16).

The SM method

Similarly equation (26) is integrated over the control volume $\left[x_{i+1/2}, x_{i-1/2}\right]$ and discretized in time by a forward finite difference method while in space by a centered finite difference:

$$\begin{aligned} \mu_{i,j}^{n+1} = \mu_{i,j}^n &- \frac{\vartheta \Delta t}{c_b \delta \Delta x}\left[\left(F_\mu\right)_{i+\frac{1}{2}}^n q_{i+\frac{1}{2}}^{n+1} - \left(F_\mu\right)_{i-\frac{1}{2}}^n q_{i-\frac{1}{2}}^{n+1}\right] \\ &- \frac{(1-\vartheta)\Delta t}{c_b \delta \Delta x}\left[\left(F_\mu\right)_{i+\frac{1}{2}}^{n-1} q_{i+\frac{1}{2}}^n - \left(F_\mu\right)_{i-\frac{1}{2}}^{n-1} q_{i-\frac{1}{2}}^n\right] + (\mu^*)_{i,j}^n \frac{\Delta z_b^{n+1}}{\delta \Delta x} \\ &= 0 \end{aligned} \tag{28}$$

Herein the values at the nodes $i + 1/2$ are computed with an upwind interpolation while $\left(F_\mu\right)_{i+1/2}^n$ is calculated through equation (21). In this preliminary work the value of the variance is considered constant in time and the concentration $c_{i+1/2}^{n+1}$ is evaluated by means of equation (24).
The advantages of this method is the decrease in the number of variables. On the other hands, no other information regarding the kind of grain-size distribution are given by the model and only the evolution of the mean and eventually of the other moments is computed. In the next section some initial results are reported.

Results

A channel of 1km with an excavation is considered in this test case. The initial condition is the stationary flow in the case of fixed bed with discharge $3.3 m^3/s$ as shown in Figure 5.
The boundary conditions at the upstream are constant: the liquid discharge ($3.3 m^3/s$) and the concentration *c* at the equilibrium. At the downstream boundary the free elevation is imposed equal to $66.4 m$. The mixing layer thickness is constant in space and time and equal to $1 cm$. The diameters are log-normally distributed with mean μ equal to $2.78 mm$ and variance σ^2 to $0.25 mm$ in either the mixing layer and the substrate.

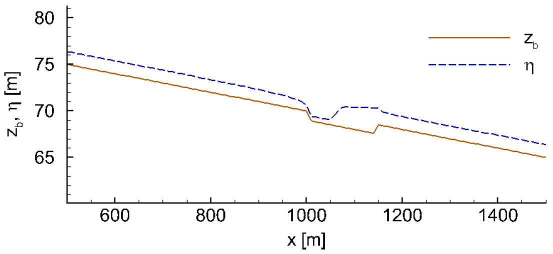

Figure 5: Initial condition: bed profile and free surface elevation.

Firstly, the test is performed with the BFM method and the distribution curve is discretized with 15 classes as shown in Figure 6.
Secondly, in the SM model the same distribution is considered Figure 7. Notice that the variance is kept constant and only the mean changes during the simulation.

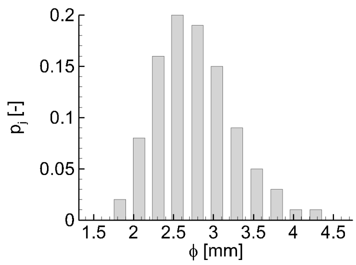

Figure 6: Probability or available fraction along the channel in the mixing layer and in the substrate at the initial condition.

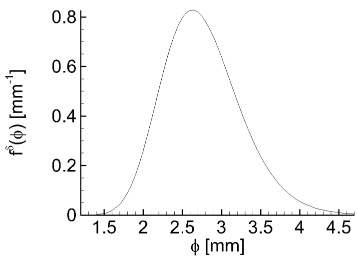

Figure 7: Density function ($\mu = 2.78mm, \sigma^2 = 0.25mm$) in the mixing layer and in the substrate at the initial condition.

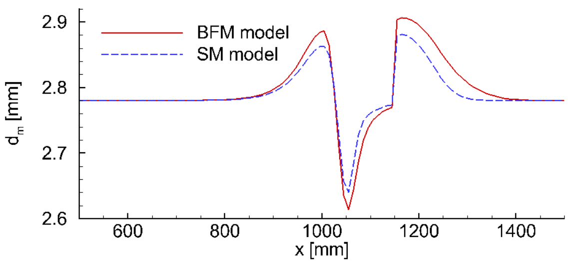

Figure 8: Comparison between the two model at 200000 s.

The profile of the mean diameter in the mixing layer resulting from the SM model is compared with the mean diameter calculated as $\sum_{j=1}^{m} p_j d_j$ for the BFM approach as shown in Figure 8.

Conclusion

The numerical results show a good agreement of the mean diameter profiles. In conclusion, the advantage of the SM approach is the lower computational cost. On the other hand more information about the grain-size distribution is given with the BFM approach. In fact, the limit of the SM model is the loss of information about the shape of the distribution curve. Better results could be obtained by adding the equation for the evolution of the variance and of the others statistical moments. In fact, the future development of this work will be derivation of the equation for the variance and its discretization in order to have a model suitable for the study of non-uniform sediment transport.

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