NUMERICAL SIMULATION OF VARIABLY SATURATED FLOW THROUGH EARTH EMBANKMENTS USING THE LATTICE-BOLTZMANN METHOD

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Abstract

An accurate determination of variably saturated seepage flow through earth embankments is an important engineering task. Numerous approaches exist solving the highly non-linear Richard's equation for this task, typically based on Finite-Element or Finite-Difference formulations. Recently, a modeling approach using the Lattice-Boltzmann method (LBM) for the Richards equation was presented and validated on simple test cases. Models based on the LBM have advantageous characteristics concerning simplicity, complex geometries, parallel efficiency and the possibility to easily extend the model framework to other flow processes, making it suitable for coupled simulations. In this work the LBM is applied for the Richard's equation in 2D and 3D on structured meshes. Different formulations of the Richard's equation based on the moisture content and the pore pressure are used. Two types of retention models after Brooks-Corey and Van Genuchten are implemented. The applicability of the approach is investigated using idealized test cases and scenarios concerning small-scale and large-scale embankments. The results are compared to selected analytical, experimental and numerical results.

Introduction

Seepage analysis is an important part of geotechnical engineering and is required for design and stability evaluations of earth embankment structures. It is well known that slope stabilities of river dikes or earth dams depend on the pore pressure distributions within the saturated and unsaturated zones of the embankment body.

Often seepage analysis is performed solving the groundwater equations for subsurface flow with a free surface. However, in such approaches flow in the unsaturated zones is neglected and it is difficult to model water infiltration into the embankment body.

In contrast, solving the Richard's equation accounts for the unsaturated zone and allows an accurate modeling of water infiltration. Constitutive models consisting of a retention curve and a relative hydraulic conductivity function allow approximating the multi-phase flow in the unsaturated zone with a single differential equation. Thereby the assumption is made that the air phase is always continuous and at atmospheric pressure, which is often said to be accurate enough for most practical applications (e.g. Lam *et al.* 1987).

The numerical solution of the Richard's equation is a challenging task due to strong non-linearities introduced by the constitutive models. Additionally, at the interfaces between different soils in heterogeneous embankments, steep jumps and abrupt changes in the variables may occur. Many models were presented in the past solving the Richard's equation based on Finite-Difference or Finite-Element methods and showed good results. A novel application of the Lattice-Boltzmann method on the Richard's equation was recently presented by Ginzburg *et al.* (2004) and Ginzburg (2006), basing on a LBM approach for generic anisotropic advection-dispersion equations. The application of the LBM has some advantages which can make it an interesting alternative choice compared to classical continuum approaches. The method is simple and easy to implement and it allows the modeling of complex geometries using bounce-back boundaries. Also, the method is local and therefore suited well for parallelization. Furthermore, it can be applied to a variety of other flow equations with rather small modifications in the modeling framework.

Ginzburg adapted solution strategies for advection-diffusion problems to different formulations of the Richard's equation, like the moisture θ formulation and mixed moisture-pressure head θ - h formulation. The method was validated on idealized test cases with simple geometries. Since no other applications of this method are known to the author up to now, there is need for additional tests and applications on realistic scenarios with more complex geometries. The applicability of the method on variably saturated flow is investigated here with a focus on flow through earth embankments. The applications concern small-scale and large-scale geometries as well as homogeneous and heterogeneous embankments. The model results are compared against analytical solutions, experimental measurements and other numerical results.

Physical description

Richard's equation

The Richard's equation is a non-linear partial differential equation, which can be formulated in form of an advection-diffusion equation. The soil moisture content θ [-] in the equation is defined as the effective water saturation $\theta = (\theta_0 - \theta_R) / (\theta_S - \theta_R)$, with θ_0 = water content, θ_R = residual water content and θ_S = saturated water content (=porosity). The other main variable is the pore pressure of the water within the embankment body which is described in a pressure head formulation as $h = p / (\rho g)$ [L].

The Richard's equation is applied in a moisture formulation for θ as primary variable as

$$\begin{aligned} &\frac{\partial\theta}{\partial t} - \nabla\ K\nabla z\ - \nabla(D\theta) = Q \\ &D = K\frac{\partial h}{\partial\theta}. \end{aligned} \tag{1}$$

It is used in a mixed moisture and pressure head formulation for θ and h as

$$\begin{aligned} &\frac{\partial\theta}{\partial t} - \nabla\ K\nabla z\ - \nabla(D\nabla h) = Q \\ &D = K, \end{aligned} \tag{2}$$

where D is a diffusivity and K is the conductivity which is calculated as $K = k_r(\theta)k_f$, being the product of the dimensionless relative conductivity k_r [-] and the hydraulic soil conductivity k_f [L/T]. These formulations of the Richard's equation are made dimensionless for the computations using the cell size Δx as length scale and $\Delta x / \Delta t$ as velocity scale, leading to a "mesh speed" of $c = 1$ in the LBM.

Following Ginzburg, the primary variable θ is used for the unsaturated zone as well as for the saturated zone. Therefore the retention curve $h = f(\theta)$, which is defined for the unsaturated zone only, is extrapolated linearly into the saturated zone as

$$h(\theta) = (\theta - 1)\left.\frac{\partial h}{\partial\theta}\right|_{\theta=1} + h_s \qquad \theta \geq 1.0, \tag{3}$$

with $\partial h/\partial\theta$ being the gradient of the retention curve at the transition to the saturated zone ($\theta = 1.0$) and h_s [L] being the air entry pressure head, at which air can enter the pores when the soil is drained. This approach has the advantage that no special treatment and no change of variables regarding the saturated/unsaturated zones are necessary. However, it leads to an artificial compressibility error in unsteady simulations. This error is neglected here, but can be reduced using sub-iterations (Ginzburg *et al.* 2004).

Retention curves

Empirical closures for the retention curve $h = f(\theta)$ and the relative permeability function $k_r = f(\theta)$ are required to solve the equation. The retention function in soil sciences describes how much water is retained in the soil by the capillary pressure and can be seen as a description of the pore distributions of the soil. The relative conductivity function describes the water mobility within the unsaturated zone depending on the moisture contents and equals 1.0 in the saturated zone. Two different empirical relationships are employed here, the approach after Brooks-Corey and Mualem (1976, BCM) and a modified version of the Van Genuchten and Mualem model (1980, VGM).

For the BCM model the following relations are used

$$\begin{aligned} &h(\theta) = h_s\theta^{-1/\lambda}, \quad k_r = (h / h_s)^{-(4\lambda+2)} \\ &\frac{\partial h}{\partial\theta} = -\frac{h_s}{\lambda}\theta^{\frac{-1}{\lambda}-1}, \end{aligned} \tag{4}$$

with λ [-] being a soil parameter. For the modified VGM model the functions and derivate are as follows

$$\begin{aligned} &h(\theta) = \frac{-1}{\alpha}\left[-\varkappa\left(\frac{\beta}{\theta}\right)\right]^{1/n}, k_r = \sqrt{\theta}\left(\frac{1 - \varkappa^m\theta / \beta}{1 - \varkappa^m 1 / \beta}\right)^2 \\ &\frac{\partial h}{\partial\theta} = \frac{1 - m}{\beta m\alpha}\frac{\theta}{\beta}^{-1/m-1} - \varkappa\left(\frac{\beta}{\theta}\right)^{-m}, \end{aligned} \tag{5}$$

with $\varkappa(x) = 1 - x^{1/m}$ and $m = 1 - 1 / n$. The constants α [1/L] and n [-] describe the soil properties. The modified Version of the VGM model is used instead of the VGM model in order to prevent infinite slopes $\partial h/\partial\theta$ at the transition to the saturated zone.

Lattice-Boltzmann Method

Introduction

The LBM is a mesoscopic modeling approach which is positioned in between microscopic, particle-based dynamics and macroscopic continuum approaches. The Boltzmann equation is formulated for a probability distribution function $f(\vec{r}, t)$ of particles in the phase-space $\vec{r}(\vec{x}, \vec{v})$. It represents the distribution of particles at time t with locations and velocities in between $\vec{r}$ and $\vec{r} + \Delta\vec{r}$. The LBM solves the Boltzmann equation in a discrete form on a uniform mesh. Mainly three computational steps are applied, whereas the first two steps are similar to particle-based approaches:

- propagation of f from cell to cell,
- collision operator of f within the cells, and
- update of the macroscopic variables.

The collision operator is approximated using the BGK (Bathnagar, Gross & Krook) approach which assumes a relaxation of f towards an equilibrium distribution function f^{eq}. The macroscopic variables of interest can finally can be derived from the distribution function f.

The LBM can be applied for the solution of different types of macroscopic governing equations by using different equilibrium functions. Important application fields are e.g. the Navier-Stokes or shallow water equations. Also, several approaches solving the advection-diffusion equation were proposed in the past (e.g. Flekkoy 1993, van der Sman & Ernst 1999, Ginzburg 2005). The present work bases on a recent adaptation of such an advection-diffusion LB scheme to the Richard's equation.

The discrete Boltzmann equation for the spatial mesh directions q reads

$$\begin{aligned} & f_q(\vec{r} + \Delta t \vec{c}_q, t + \Delta t) = \\ & \qquad f_q(\vec{r}, t) + \omega \left[f_q(\vec{r}, t) - f_q^{eq}(\vec{r}, t) \right] + Q_q, \end{aligned} \tag{6}$$

with a single time relaxation parameter ω for the collision operator, being determined as a function of the diffusivity D as $\omega = -1.0 / (D / c_s + 0.5)$. The diffusivity is determined for the θ formulation as $D = k_r(\theta) k_s \partial h / \partial \theta$ and for the mixed θ - h formulation as $D = k_r(\theta) k_s$. The variable Q_q on the right hand side is an external source for modeling water infiltration into the embankment.

The uniform mesh is constructed with quadratic cells in 2D or cubic cells in 3D using a set of q discrete velocities c_q which connect the grid cells with each other. In 2D simulations a mesh with 9 different directions is used (2DQ9) and in 3D simulations 15 different directions are used (3DQ15). The directions of the q discrete velocities (see Figure 1) are set in 2D and 3D as

$$\vec{e} = \begin{cases} (0,0), (\pm 1, 0), (\pm 1, \pm 1), (0, \pm 1) & (D2Q9) \\ (0,0,0), (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1), (\pm 1, \pm 1, \pm 1) & (D3Q15) \end{cases} \tag{7}$$

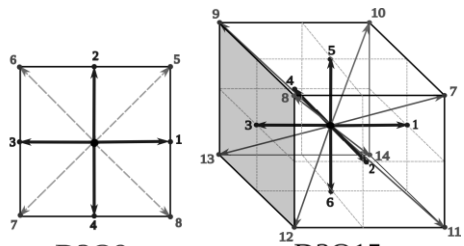

Figure 1: 2D lattice with 9 and 3D lattice with 15 directions

Equilibrium functions

The equilibrium function for the θ formulation is given in first order accuracy by

$$f_q^{eq} = \begin{cases} 1.0 - \varphi c_s^2 \cdot \theta & q = 0 \\ t_q \cdot \theta c_s^2 + \vec{I} \cdot \vec{c} & q = 1..n \end{cases}, \tag{8}$$

where $c_s^2 = c / \vartheta$ with ϑ being a free constant. The constant φ is 5/3 for 2D simulations and 7/3 for 3D simulations. The method is local because the equilibrium function needs no information from neighboring cells. This is an advantageous property especially regarding the parallelization of the code.

Accordingly, the equilibrium function for the mixed θ - h formulation is

$$f_q^{eq} = \begin{cases} \theta - \varphi c_s^2 \cdot h & q = 0 \\ t_q \cdot h c_s^2 + \vec{I} \cdot \vec{c} & q = 1..n \end{cases}. \tag{9}$$

In contrast to the θ formulation, the mixed θ - h formulation is able to reproduce the continuous transition of the pressure head at the interface of different soils. Therefore, the mixed θ - h formulation should be applied in cases of heterogeneous embankments with core and filter zones. The θ formulation, however, has advantageous stability conditions for imbibition problems (Ginzburg 2006) and as such is recommended for use in case of homogeneous embankments.

The weighting factors t_q for the lattice directions q can be derived for the chosen lattice configuration. The values for the 2DQ9 and 3DQ15 models are given in Table 1.

Table 1: Lattice weighting coefficients for directions q

q	t_q (D2Q9)	q	t_q (D3Q15)
1-4	1/3	*1-6*	1/3
5-8	1/12	*7-14*	1/24

To calculate the equilibrium functions given above, the advective, gravitational term $\vec{I}$, which acts in vertical downward direction, is needed and can be evaluated as

$$\vec{I} = -k_r(\theta) k_s \vec{e}_z. \tag{10}$$

Derivation of macroscopic values

Using these relationships and the retention curves, the Boltzmann equation is solved by applying the propagation and collision steps mentioned above. The update step, i.e. the derivation of the macroscopic variables θ, h and the

Darcy velocity v_f from the distribution functions f_q, can finally be done with following relations:

$$\theta = \sum_{q=0}^{n} f_q \quad , \quad p = f(\theta)$$
$$\vec{v}_f = (\theta_s - \theta_r) \cdot \left[\sum_{q=0}^{n} \vec{c} f_q^{eq} + \begin{pmatrix} c_{x0} & \cdots & c_{xn} \\ c_{y0} & \cdots & c_{yn} \end{pmatrix} \cdot \begin{pmatrix} f_0^{eq} - f_0 \\ \vdots \\ f_n^{eq} - f_n \end{pmatrix} \right]. \tag{11}$$

Boundary and initial conditions

At the mesh boundaries the values of the distribution function f_q in the incoming directions are unknown and must be provided.

- For solid walls standard bounce-back boundaries are used. The unknown incoming distribution functions f_q thereby are set equal to the outgoing values to simulate wall reflection. Using this type of boundary condition allows incorporating even complex boundaries. It is thus possible to easily integrate piling walls or impervious zones and it seems suitable for embankment breach simulations with complex and dynamically changing geometries.
- A water column above the embankment is modeled using a pressure boundary, thereby presuming a hydrostatic pressure distribution. Equilibrium conditions are assumed at the boundary which allows calculating the values for the incoming directions (f_q = f_q^{eq}).
- The seepage flow out of the embankment is modeled with a combined approach. In the saturated zone ($\theta \geq 1.0$) a constant saturation of 1.0 is set at the boundary cells. In the unsaturated zone ($\theta < 1.0$) a bounce-back or zero gradient boundary is used. The zero gradient boundary sets the incoming values of f_q to the values of the corresponding neighboring cell.

The exact treatment of sloped or curved boundaries may become difficult, especially in 3D. Here, for simplicity, the sloped embankment faces are approximated using a series of steps using reflection angles of 0°, 45° or 90°. These simplifications can reduce the numerical accuracy in the vicinity of the embankment faces. More accurate boundary treatment schemes can be implemented alternatively (e.g. Ginzburg & d'Humières 1996, Mei *et al.* 1999) if higher accuracies are needed.

As initial conditions, the pore pressures or saturations in the domain can be given. The initial distribution functions are then set equal to the corresponding equilibrium values (f_q = f_q^{eq}) assuming equilibrium conditions.

Tests & Applications

Confined, saturated flow

As a first test case a steady-state subsurface flow on a quadratic mesh of 1x1 m is selected with a cell size of Δx = 0.01 m. The soil's hydraulic conductivity is set to $k_{f,w}$ = 0.001 m/s in the west part of the domain and to $k_{f,\mathrm{e}}$ = 5E−4 m/s in the east part of the domain. At the west boundary a hydrostatic pressure head of h_w = 5 m is set and at the east boundary h_{e} = 2 m. The other boundary cells are treated as bounce-back boundaries. The constant ϑ is set to 3. This configuration results in a confined subsurface flow with constant pressure gradients and velocity field. The obtained results are illustrated in Figure 2 (left).

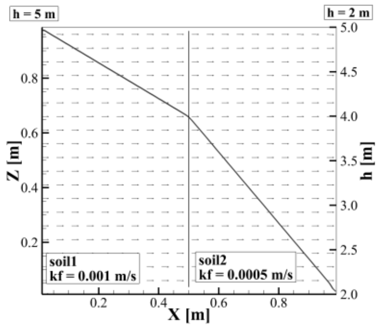

Figure 2: Saturated flow through layered domain (left).

Using the mixed θ - h formulation the model is able to correctly reproduce the continuous transition of the pressure at the interface of the two soils. The pressure gradients as well as the velocities are constant and equal the analytical solutions (v_f = 0.002 m/s, $\partial h / \partial x_w$ = −2.0, $\partial h / \partial x_{\mathrm{e}}$ = −4.0).

Soil infiltration

The second test case investigates unsteady flow within the unsaturated zone. In this example the downward infiltration of water into a clay soil is considered using a 1x1 m domain with Δx = 0.01 m and Δt = 1.0 s. This example was investigated with a FE-model (HYDRUS) in the work of Vogel *et al.* (2001) and was already used by Ginzburg *et al.* (2004) as validation example for the LBM approach. Therefore this test case is not described in detail here.

The propagation of the infiltration front is reproduced well using this approach. Comparisons with the results obtained by the FE-model show negligible differences and confirm the results previously obtained by Ginzburg.

Flow through small-scale embankment

In this setup the 3D unsteady seepage flow through a laboratory dike is modeled. The experimental investigations were made at the TU Berlin (see Pham Van (2009) for details). The homogeneous dike is 0.6 m high, 4.0 m long and 0.4 m wide. The sand dike material has a hydraulic conductivity of k_f = 9.5E−4 m/s[1], a saturation moisture content of θ_S = 0.49 and a residual water content of θ_R = 0.01. As initial state the measured water content of θ_0 = 0.115 is applied and the air entry pressure is set to h_s = −0.035 m. The computational time step is set to Δt = 0.5 s and Δx = 0.01 m. At the upstream embankment slope a time dependent pressure boundary is applied, according to the rising water level in the experiment. At the downstream embankment slope a seepage boundary is set. The VGM-model is used with α = 14.5 1/m and n = 2.68 for sand material. This and the following VGM parameters were chosen with respect to the listed values in Vogel *et al.* (2001). In this case the VGM model shows to be more stable than the BCM model. Since the dike is homogeneous, the θ formulation is used with ϑ = 30.

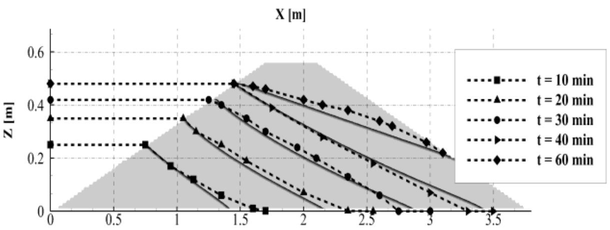

Figure 3: Cross sectional view of measured (dashed) and simulated (bold) temporal development of seepage front.

Measured and simulated results are depicted in Figure 3. The temporal development of the saturation front is in good accordance between measurement and simulation throughout time and the model runs stable.

Flow through large-scale embankments

To test the model on large-scale scenarios, two model configurations are chosen from Bowles (1994). These test cases concern steady state conditions in a homogeneous and a heterogeneous embankment and were recommended as tests for variably saturated seepage modeling in Chapuis *et al.* (2001).

Homogeneous embankment

The first test in Bowles (9-5a) regards an embankment of 100 m length, 20 m height and a crest length of 10 m ranging from x = 50-60 m (Fig. 4). The embankment is made of homogeneous material with a hydraulic soil conductivity of k_f = 6.67E−6 m/s and Δx = 0.25 m. A constant water level is set at the upstream embankment slope of h = 18.5 m and a seepage boundary condition at the downstream embankment slope. The VGM model is applied (α = 5.14 1/m, n = 1.69), which shows again a more stable behavior than the BCM model. The air entry pressure is h_s = −0.05 m and the θ formulation is used with a value of ϑ = 100.

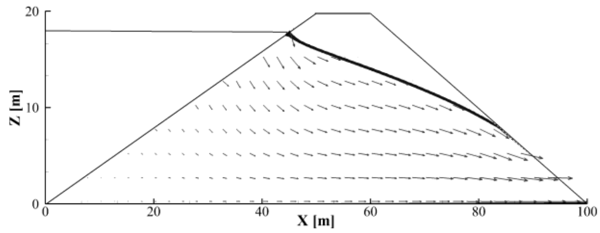

Figure 4: Steady seepage flow through homogeneous, large-scale embankment (9-5a in Bowles).

The total specific seepage rate per unit width at the downstream embankment slope was estimated by Bowles to Q = 2.13E−5 m^2/s, neglecting flow in the unsaturated zone. Numerical simulations with FE models led to values of Q = 2.35E−5 m^2/s (Crespo *et al.* 1993) and Q = 2.28E−5 m^2/s (Chapuis & Aubertin 2001). Using the LBM, a specific discharge of Q = 2.34E−5 m^2/s is obtained which compares well to the previous investigations. The water table reaches the air side of the embankment at an elevation of about 8.0 m which lies in between the 6.5 m of Bowles (1994) and the 9.0 m of Chapuis & Aubertin (2001).

Heterogeneous embankment

The second test in Bowles (9-5b) regards a heterogeneous embankment of 190 m length, 45 m height and a crest length of 10 m ranging from x = 90-100 m. The dam material has a hydraulic conductivity of k_f = 2E−7 m/s and a filter zone is installed at the right embankment toe with a hydraulic conductivity of k_f = 1E−4 m/s. The cell size is set to Δx = 0.5 m and h_s = −0.03 m. A pressure boundary with a constant water level is set at the upstream embankment slope of h = 40 m and a seepage boundary condition at the downstream embankment slope. The other boundary cells are treated as bounce-back boundaries. Again the VGM retention model is applied (embankment material: α = 2.0 1/m, n = 1.41; filter material: α = 14.5 1/m, n = 2.68). Here, the mixed θ-h formulation is chosen since the embankment is heterogeneous and a constant ϑ = 1000 is used.

The pressure continuity at the interface between the different soil materials is correctly reproduced (Figure 5).

[1]The value given in the reference of k_f = 0.95E−4 m/s is supposed to be a typo.

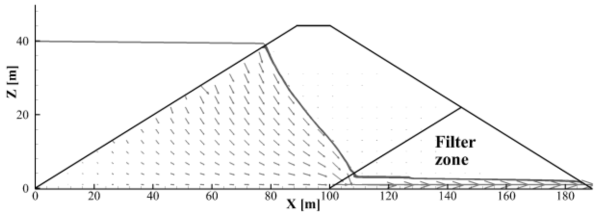

Figure 5: Steady seepage flow through heterogeneous large-scale embankment with toe filter (9-5b in Bowles).

Further, the seepage line and the pressure distribution qualitatively fit the results obtained in the references. The modeled specific leakage flow rate per unit width is Q = 3.4E−6 m^2/s, which is in a reasonable range compared to the flow rates of Q = 3.8E−6 m^2/s (Bowles 1994), Q = 5.1E−6 m^2/s (Crespo *et al.* 1993) and Q = 4.23E−6 m^2/s (Chapuis *et al. 2001*[2]).

Conclusions and outlook

In this work a LBM for the Richard's equation was implemented and investigated. Ginzburg's derivations and approaches using θ and θ - h formulations were applied in 2D / 3D in combination with retention models of BCM and VGM type. The model was successfully validated against several test cases confirming the approach and the assumptions made. Model applications on scenarios of small-scale and large-scale embankments led to good results and were stable over a wide range of parameters. The θ formulation for homogeneous embankments and the VGM retention model thereby showed a more stable behavior than the mixed θ - h formulation and the BCM model. The method is able to deal with realistic geometries and seems suitable also for practical engineering applications. However, additional tests of the approach should be made for realistic scenarios to further investigate possible strengths and limitations.

Enhancements of the model could be a consideration of higher order accuracies, sub-iterations, anisotropic conditions and more accurate representations of sloped or curved boundaries. To further improve the stability for heterogeneous embankments, the implementation of a variable switching procedure could be an option, whereas the θ - h formulation is applied in areas with changing soil properties and the θ formulation otherwise.

As further steps, the coupling with an embankment breach model and the integration into the software BASEMENT (Faeh *et al.* 2011) are planned. An advantageous property of the method thereby is its suitability for complex and dynamically changing geometries due to the use of bounce-back boundaries. Furthermore, the applied LBM framework can in principle be extended to solve also surface water flow around or over the embankment structure. This provides a promising methodology for coupled subsurface-surface flow simulations.

Acknowledgments

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References

Bowles, J. E. (1984). *Physical and geotechnical properties of soils*. 2nd Edition, McGraw-Hill, New York.

Brooks, R. H., & Corey, A. T. (1964). *Hydraulic properties of porous media.* Hydraulic paper, 3, Colorado State University.

Chapuis, R. P, & Aubertin, M. (2001). *A simplified method to estimate saturated and unsaturated seepage through dikes under steady-state conditions.* Canadian Geotechnical Journal, 38, pp. 1321-1328.

Chapuis, R. P., Chenaf, D., Brussière, B., Aubertin, M. & Crespo, R. (2001). *A user's approach to assess numerical codes for saturated and unsaturated seepage conditions*. Canadian Geotechnical Journal, 38, pp. 1113-1126.

Crespo, R (1993). *Modélisation des écoulements à travers les ouvrages de retenue et de confinement des résidus miniers par un logiciel d'éléments finis.* Masters thesis, École Polytechnique de Montréal.

Faeh, R., Mueller, R., Rousselot, P., Vetsch, D., Vonwiller, L., & Volz, C. (2011). *BASEMENT – Basic Simulation Environment for Computation of Environmental Flow and Natural Hazard Simulation*, online program manual, VAW, ETH Zurich, www.basement.ethz.ch.

Flekkoy, E. G. (1993). *Lattice Bhatnagar-Gross-Krook models for miscible fluids.* Physical Review E, 47, pp. 4247-4257.

Ginzburg, I. (2005). *Equilibrium type and link-type lattice Boltzmann models for generic advection and anisotropic-dispersion equation.* Advances in Water Resources, 28, pp. 1171-1195.

Ginzburg, I. (2006). *Variably saturated flow described with the anisotropic Lattice Boltzmann methods.* Computers & Fluids, 35, pp. 831-848.

Ginzburg, I., Carlier, J., & Kao, C. (2004). *Lattice Boltzmann approach to Richard's equation.* Proceedings of the CMWR XV, CT Miller, pp. 583-597, Chapel Hill, NC, USA.

Ginzburg, I., & d'Humières, D. (1996). *Local Second-Order Boundary Methods for Lattice Boitzmann Model.* Journal of Statistical Physics, 84, pp. 927-971.

Lam, L., Fredlund, D. G., & Barbour, S. (1987). *Transient seepage model for saturated-unsaturated soil systems: a geotechnical engineering approach*. Canadian Geotechnical Journal, 24, pp. 565-580.

Mei, R., Luo, L. & Shyy, W. (1999). *An Accurate Curved Boundary Treatment in the Lattice Boltzmann Method.* Journal of Computational Physics, 155, pp. 307-330.

Mualem, Y. (1976). *A new model for predicting the hydraulic conductivity of unsaturated soils*. Water Resources Research, 12(3), pp. 513-52.

Pham Van, S. (2009). *Application of different model concepts for simulation of two-phase flow processes in porous media with fault zones.* Ph.D Thesis, Book Series Volume 3, TU Berlin.

van Genuchten, M. T. (1980). *A closed-form equation for predicting the hydraulic conductivity of unsaturated soils*. Soil Science Society of America Journal, 44, pp. 892-898.

van der Sman, R. G. M., & Ernst, M. H. (1999). *Convection-diffusion lattice Boltzmann scheme on a orthorhombic lattice*. Journal of Computational Physics, 160, pp. 766-782.

Vogel, T., van Genuchten, M. T., & Cislerova, M. (2001). *Effect of the shape of the soil hydraulic functions near saturation on variably saturated flow predictions.* Advances in Water Resources, 24, pp. 133-144.

[2] The unit given in the reference of m^2/min should read m^2/s.