A GENERAL CLOSURE ROE SOLVER FOR HYPER-CONCENTRATED SHALLOW FLOWS OVER MOBILE BED

Lorenzo Begnudelli[1] & Giorgio Rosatti[1]

[1]CUDAM (Centro Universitario per la Difesa Idrogeologica dell'Ambiente Montano),
University of Trento, Italy, Via Mesiano 77, 38123 Trento
E-mail: lorenzo.begnudelli@unitn.it, giorgio.rosatti@unitn.it

Abstract

In this paper we present a novel general formulation of the Generalized Roe solver for hyper-concentrated 1D shallow flows over a mobile bed.
Hyper-concentrated flows are mathematically defined by a hyperbolic system of three partial differential equations. The system shows non-conservative terms, is highly nonlinear, and its whole structure depends on the closure relationship used to define the concentration.
In earlier works, a well-balanced Generalized Roe solver has been derived for 1D and 2D flows. In these approaches, the solution of the Riemann Problem (RP) is obtained from the exact solution of a locally linearized problem, by writing a Jacobian matrix of the system as a function of proper averages of the primitive variables. The formulation of the scheme is not unique and depends on the adopted set of averages. Based on the closure, not only different formulations of the solver are obtained, but more in general the derivation can be quite easy or extremely complicated. In any case, so far only Roe schemes relative to specific closures have been derived.
In this paper, we write a general formulation of the Roe scheme, valid for any possible closure. In fact, we treat the concentration as a function of the other variables and write the Jacobian in terms of its partial derivatives. The method is completely general, easy to implement, and as accurate as the standard Roe approach.

Introduction

In mountain regions, water flows are often associated with heavy sediment movements that can generate significant erosion or deposition. Examples of hyper-concentrated flows are debris flows and dam-breaks over mobile bed. More in general, in mountain environment river flows are often connected with important sediment movements.
Clearly, the capability of modeling and forecasting these events is therefore a key issue for the safety of mountain regions. However, the mathematical description of these phenomena is particularly challenging due to the properties of the system of governing equations. The problem is mathematically defined by a hyperbolic system of three partial differential equations system that shows non conservative terms and highly nonlinear relations between primitive and conserved variables. The concentration of solid phase c not only is present in the continuity equations of solid and mixture mass, but also plays a role in the momentum equations, because in the case of hyper-concentrated flows its contribution to the mixture momentum is not negligible. Thus, the structure of the entire system is highly dependent on the closure relationship used to define the concentration, i.e. to the chosen rheological model.
Among the several recent contributions on this account, particularly relevant to this paper is the work of Rosatti et al. (2008), where a new Generalized Roe (GR) scheme has been introduced in the numerical modelling of 1D, two-phase shallow flows. More recently, Rosatti and Begnudelli (2011) have extended the GR scheme to the 2D case and applied it to the numerical model Trent2D (Armanini et al. (2009). Also, Murillo & García-Navarro (2010) have used the Roe scheme within an Exner-based coupled model for two-dimensional transient flow over erodible bed.
The Roe scheme has been proven to be very accurate, but his main drawback is that the formulation of the scheme depends strongly on the closure relationships adopted in the model, e.g. the sediment transport formula. So far, only Roe schemes relative to specific closures have been derived.
In this paper, this problem is overcome by introducing a general formulation of the Roe scheme, valid for any possible closure.
The paper is structured as follows: first we present the mathematical model, then we describe the classical Generalized Roe approach and the new general closure Roe solver, lastly we compare the results of the two approaches both analytically and numerically.

The mathematical model

The mathematical model is constituted by the depth-integrated, shallow-water conservation equations of solid mass, mixture mass and mixture momentum for a two-phase flow over a mobile-bed. To derive the equations, the following assumptions are introduced: inter-phase forces due to differences between solid and liquid phase velocities are negligible; the pressure distribution is linear along the vertical direction; the concentration is constant through the flow depth; tangential stresses are present only at the bed (see Armanini et al. (2009) for more details).

The resulting mathematical model for the 1D case is described by the following system:

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x}+\mathbf{H}\frac{\partial \mathbf{W}}{\partial x}=\mathbf{T} \qquad (1)$$

where the vector of conserved variable **U** and the conservative fluxes **F** and **G** are defined as:

$$\mathbf{U}=\begin{bmatrix} h+z_b \\ ch+c_b z_b \\ c^{\delta} uh \end{bmatrix} \qquad \mathbf{F}=\begin{bmatrix} uh \\ cuh \\ c^{\delta}(u^2 h+gh^2/2) \end{bmatrix} \qquad 2)$$

where h is the mixture depth, z_b is the bed elevation, c is the mixture concentration, c_b is the sediment concentration in the bed (constant), u is the depth-averaged mixture velocity and c^{δ} = $(1+\Delta_s c)$, where $\Delta_s=(\rho_s-\rho_w)/\rho_w$, being ρ_s and ρ_w the densities of the liquid and solid phase respectively.

H ∂**W**/∂x is the non conservative term deriving from the pressure exerted by the bed on the control volume, where $\mathbf{W}=(h,z_b,u)^T$ is the vector of the primitive variables and:

$$\mathbf{H}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & H_{33} \end{bmatrix}; \qquad (3)$$

where $H_{33}=c^{\delta}gh$ and g is the gravitational acceleration.

As for the concentration c, we consider a generic closure, which can be written in the following form:

$$c=c(u,h) \qquad (4)$$

Finally, the source term **T** is a vector of the type: $[0,0,-\tau/\rho_w]$, where τ is the tangential bed stress whose expression of the depends on the closure used for the phenomenon under investigation. Here, we will focus on the homogeneous part of the system only.

The Riemann Problem

We recall here the general formulation of the Roe's method. We look for the general solution of the following problem:

$$\begin{cases} \dfrac{\partial \mathbf{U}}{\partial t}+\dfrac{\partial \mathbf{F}}{\partial x}+\mathbf{H}\dfrac{\partial \mathbf{W}}{\partial x}=\mathbf{0} \\ \mathbf{U}(x,0)=\begin{cases} \mathbf{U}_L & if \quad x<0 \\ \mathbf{U}_R & if \quad x\geq 0 \end{cases} \end{cases} \qquad (5)$$

The Generalized Roe Solver (Rosatti et. al., 2008) approximates (5) with the following linear RP:

$$\begin{cases} \dfrac{\partial \mathbf{U}}{\partial t}+\tilde{\mathbf{J}}(\mathbf{U}_L,\mathbf{U}_R)\dfrac{\partial \mathbf{U}}{\partial x}=\mathbf{0} \\ \mathbf{U}(x,0)=\begin{cases} \mathbf{U}_L & if \quad x<0 \\ \mathbf{U}_R & if \quad x\geq 0 \end{cases} \end{cases} \qquad (6)$$

where $\tilde{\mathbf{J}}$ ($\mathbf{U}_L$,$\mathbf{U}_R$) is a suitable matrix whose value depends on the left and right initial conditions and can be obtained by imposing the following constraints:

$$\tilde{\mathbf{J}}(\mathbf{U}_L,\mathbf{U}_R)=(\tilde{\mathbf{A}}+\tilde{\mathbf{H}})\tilde{\mathbf{B}}^{-1} \qquad (7)$$

where:

$$\tilde{\mathbf{B}}(\mathbf{W}_L-\mathbf{W}_R)=\mathbf{U}_L-\mathbf{U}_R \qquad (8a)$$

$$\tilde{\mathbf{A}}(\mathbf{W}_L-\mathbf{W}_R)=\mathbf{F}_L-\mathbf{F}_R \qquad (8b)$$

$\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ can be determined as Jacobian matrices with respect to the primitive variables (h, z_b, u), evaluated for proper averages ($\tilde{h}$, $\tilde{z}_b$, $\tilde{u}$) of the left and right variables.

The Generalized Roe numerical flux $\mathbf{F}^{GR}$ is found as:

$$\mathbf{F}^{GR}=\mathbf{F}_L+\sum_{m=1}^{4}(\lambda^{-}\mu\mathbf{R})^m=\mathbf{F}_R-\sum_{m=1}^{4}(\lambda^{+}\mu\mathbf{R})^m \qquad (9)$$

A Specific Closure Riemann Solver (SCRS)

We consider now the case of the 1D two-phase flows over mobile bed presented earlier and described by Eqs. (1)-(4). Clearly, the system depends strongly on the closure chosen for the concentration, as c is present both in the second and third equations. Following Rosatti (2008) and Armanini (2009) we assume the following closure relationship:

$$c=c(u,h)=\beta c_b\frac{u^2}{h} \qquad (10)$$

where β is a dimensionless transport parameter.

Such a closure has also been implemented in the 2D model TRENT2D (Armanini 2009) and widely applied to practical cases with good results. It has a physically based structure, and (also quite important for practical applications) it is particularly advantageous from the computational point of view because, when plugged into the model equations makes the resulting system of equations quite manageable. In fact, plugging Eq.(10) into Eqs. (1)-(3), we obtain the following vectors $\underline{\mathbf{U}}$ and $\underline{\mathbf{F}}$ (underline will be always used with reference to SCRS method):

$$\underline{\mathbf{U}}=\begin{bmatrix} h+z_b \\ \beta c_b u^2+c_b z_b \\ uh+c_b\beta\Delta_s u^3 \end{bmatrix} \qquad \underline{\mathbf{F}}=\begin{bmatrix} uh \\ c_b\beta u^3 \\ (1+c_b\beta\Delta_s u^3)(u^2h+gh^2/2) \end{bmatrix} \tag{11}$$

The resulting Jacobians $\partial\underline{\mathbf{U}}/\partial\mathbf{W}$ and $\partial\underline{\mathbf{F}}/\partial\mathbf{W}$ become:

$$\frac{\partial\underline{\mathbf{U}}}{\partial\mathbf{W}}=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2\beta c_b u & c_b \\ u & h+3\Lambda u^2 & 0 \end{bmatrix} \tag{12}$$

$$\frac{\partial\underline{\mathbf{F}}}{\partial\mathbf{W}}=\begin{bmatrix} u & 0 & 0 \\ 0 & 3c_b\beta u^2 & 0 \\ gh+u^2+\Lambda gu^2/2 & 2uh+\Lambda u(gh+4u^2) & 0 \end{bmatrix} \tag{13}$$

where $\Lambda=c_b\,\beta\Delta_s$

In order to find the solution of the RP, we need to find the matrices **A** and **B** that fulfill conditions (8a-b). We obtain:

$$\underline{\widetilde{\mathbf{B}}}=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2\tilde{u}\beta c_b & c_b \\ \tilde{u} & \tilde{h}+3\Lambda\overline{u^2} & 0 \end{bmatrix} \tag{14}$$

$$\underline{\widetilde{\mathbf{A}}}=\begin{bmatrix} \tilde{u} & \tilde{h} & 0 \\ 0 & 3\beta c_b\overline{u^2} & 0 \\ g\tilde{h}+\widetilde{u^2}+g\widetilde{u^2}/2 & 2\tilde{u}\tilde{h}+\Lambda\tilde{u}(g\tilde{h}+4\widetilde{u^2}) & 0 \end{bmatrix} \tag{15}$$

where:

$$\tilde{h}=\frac{1}{2}(h_L+h_R) \qquad \tilde{u}=\frac{1}{2}(u_L+u_R) \tag{16}$$

$$\widetilde{u^2}=\frac{1}{2}(u_L^2+u_R^2) \qquad \overline{u^2}=\frac{1}{3}(u_L^2+u_R^2+u_L u_R) \tag{17}$$

The derivation of the matrices $\underline{\widetilde{\mathbf{A}}}$ and $\underline{\widetilde{\mathbf{B}}}$ has been made possible by the simple structure of the matrices $\underline{\mathbf{U}}$ and $\underline{\mathbf{F}}$ resulting from the chosen closure (10). Using a different closure would make the task extremely complicated, and often impossible. Examples of different closures that one may want to adopt are sediment transport formulae as Meyer-Peter & Müller (1948) or Ashida & Michiue (1972), characterized by a threshold under which there is no sediment transport, which makes them even more difficult to be handled. Also, dealing with debris flows, different a rheological closure have been proposed in the literature (Armanini 2011).

Using any of these closures, the derivation of the matrices $\underline{\widetilde{\mathbf{A}}}$ and $\underline{\widetilde{\mathbf{B}}}$ would not be possible. For this reason, a different and more general approach has been developed, and it will be described in the next paragraph.

A General Closure Riemann Solver (GCRS)

Here, a new method for the derivation of the Riemann matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ for any generic closure, will be presented. This approach has been named General Closure Riemann Solver (GCRS), as opposed to the Specific one (SCRS) already described, and allows to compute the matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ and thus evaluate the numerical fluxes using any generic closure given by an explicit relation $c=\psi(u,h)$ or even by an implicit relation $\Psi(c,u,h)=0$.

We simply assume that ψ or Ψ is continuous, smooth, and that ψ or Ψ relates c to values of (h,u) at the same time and position (immediate adaptation hypotesis). Under these assumptions, we can write $\partial\mathbf{U}/\partial\mathbf{W}$ and $\partial\mathbf{F}/\partial\mathbf{W}$ as:

$$\frac{\partial\mathbf{U}}{\partial\mathbf{W}}=\begin{bmatrix} 1 & 0 & 1 \\ c+h\dfrac{\partial c}{\partial h} & h\dfrac{\partial c}{\partial u} & c_b \\ uc^\delta+\Delta_s q\dfrac{\partial c}{\partial h} & hc^\delta+\Delta_s q\dfrac{\partial c}{\partial u} & 0 \end{bmatrix} \tag{18}$$

$$\frac{\partial\underline{\mathbf{F}}}{\partial\mathbf{W}}=\begin{bmatrix} u & h & 0 \\ u\left(c+h\dfrac{\partial c}{\partial h}\right) & h\left(c+u\dfrac{\partial c}{\partial u}\right) & c_b \\ (gh+u^2)c^\delta+\Delta_s\Phi\dfrac{\partial c}{\partial h} & 2huc^\delta+\Delta_s\Phi\dfrac{\partial c}{\partial u} & 0 \end{bmatrix} \tag{19}$$

where:

$$q=uh \quad ; \quad \Phi=\frac{1}{2}gh^2+hu^2 \tag{20}$$

Now, we must find two matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ that satisfy Eq. (8a-b) for any left and right values of **W**, **U** and **F**. It is important to note that there is not, in general, only one way to derive the Jacobians, and that the derivation depends on how the averages are defined. Therefore, we first define a set of averages and then derive the matrices and the other averages as a consequence.

Jacobian Matrix $\widetilde{\mathbf{B}}$

- First, we define $\tilde{h}$ and $\tilde{u}$ as in Eq. (16)
- Regarding the concentration c, we consider the value corresponding to the averages $\tilde{h}$ and $\tilde{u}$ (with must not be confused with $\tilde{c}$, defined later):

$$c_\sim=c(\tilde{h},\tilde{u})=\left[\psi(h,u)\right]_{(\tilde{h},\tilde{u})} \quad ; \quad c_\sim^\delta=(1+\Delta_s c_\sim) \tag{21}$$

- As for the partial derivatives of c, we define:

$$\widetilde{\frac{\partial c}{\partial h}}=\left[\frac{\partial c}{\partial h}\right]_{(\tilde{h},\tilde{u})} \quad ; \quad \widetilde{\frac{\partial c}{\partial u}}=\left[\frac{\partial c}{\partial u}\right]_{(\tilde{h},\tilde{u})} \tag{22}$$

Having introduced these averages, we can define $\widetilde{\mathbf{B}}$ as:

$$\widetilde{\mathbf{B}}=\begin{bmatrix} 1 & 0 & 1 \\ c_{\sim}+\hat{h}\widetilde{\dfrac{\partial c}{\partial h}} & \hat{h}\widetilde{\dfrac{\partial c}{\partial u}} & c_b \\ \tilde{u}c_{\sim}^{\delta}+\Delta_s\hat{q}\widetilde{\dfrac{\partial c}{\partial h}} & \tilde{h}c_{\sim}^{\delta}+\Delta_s\hat{q}\widetilde{\dfrac{\partial c}{\partial u}} & 0 \end{bmatrix} \tag{23}$$

where the values of $\hat{h}$, $\hat{q}$ are unknowns to be defined based on Eq. (10a). In particular, considering the 2nd and 3rd equations of system (8a), we have two unknowns ($\hat{h}$, $\hat{q}$) and two equations, and the problem is thus well-posed. Solving for $\hat{h}$, $\hat{q}$ we get:

$$\hat{h}=\frac{\Delta^h(\tilde{c}-c_{\sim})+\tilde{h}\Delta^c}{\Delta^h\widetilde{\dfrac{\partial c}{\partial h}}+\Delta^u\widetilde{\dfrac{\partial c}{\partial u}}} \quad ; \quad \hat{q}=\frac{\Delta^q(\tilde{c}-c_{\sim})+\tilde{q}\Delta^c}{\Delta^h\widetilde{\dfrac{\partial c}{\partial h}}+\Delta^u\widetilde{\dfrac{\partial c}{\partial u}}} \tag{24}$$

where the averages $\tilde{q}$, $\tilde{c}$ are:

$$\tilde{q}=\frac{1}{2}(q_L+q_R) \qquad \tilde{c}=\frac{1}{2}(c_L+c_R) \tag{25}$$

and the differences Δ^h, Δ^u, Δ^q, Δ^c are generally defined as:

$$\Delta^a=a_R-a_L \tag{26}$$

Jacobian Matrix $\widetilde{\mathbf{A}}$

- Averages $\tilde{h}$, $\tilde{u}$ are defined in Eq. (16); $\widetilde{u^2}$ in Eq. (19); $\tilde{q}$, $\tilde{c}$ in Eq. (25); partial derivatives of c in Eq. (22). The term $\widetilde{\Phi}$ is defined as:

$$\Phi=\frac{1}{2}(\Phi_L+\Phi_R) \tag{27}$$

Now, we can define $\widetilde{\mathbf{A}}$ as:

$$\widetilde{\mathbf{A}}=\begin{bmatrix} \tilde{u} & \tilde{h} & 0 \\ \tilde{u}c_{\sim}+\hat{q}\widetilde{\dfrac{\partial c}{\partial h}} & \tilde{h}c_{\sim}+\hat{q}\widetilde{\dfrac{\partial c}{\partial u}} & 0 \\ \left(g\tilde{h}+\widetilde{u^2}\right)c_{\sim}^{\delta}+\Delta_s\hat{\Phi}\widetilde{\dfrac{\partial c}{\partial h}} & 2\tilde{h}\tilde{u}c_{\sim}^{\delta}+\Delta_s\hat{\Phi}\widetilde{\dfrac{\partial c}{\partial u}} & 0 \end{bmatrix} \tag{28}$$

where $\hat{q}$, $\hat{\Phi}$ are found based on Eq. (8b). Solving the 2nd and 3rd equations of the system (8b), we get:

$$\hat{q}=\frac{\Delta^q(\tilde{c}-c_{\sim})+\tilde{q}\Delta^c}{\Delta^h\widetilde{\dfrac{\partial c}{\partial h}}+\Delta^u\widetilde{\dfrac{\partial c}{\partial u}}} \quad ; \quad \hat{\Phi}=\frac{\Delta^{\Phi}(\tilde{c}-c_{\sim})+\widetilde{\Phi}\Delta^c}{\Delta^h\widetilde{\dfrac{\partial c}{\partial h}}+\Delta^u\widetilde{\dfrac{\partial c}{\partial u}}} \tag{29}$$

where $\Delta^{\Phi}=\Phi_R-\Phi_L$.

Note that $\hat{q}$ has the same value previously found (which was not obvious) and that the expressions of $\hat{h}$, $\hat{q}$ and $\hat{\Phi}$ have the same structure. In particular we have for the three terms the following singularity condition (SC):

$$\text{(SC 1)} \quad \Delta^h\widetilde{\frac{\partial c}{\partial h}}+\Delta^u\widetilde{\frac{\partial c}{\partial u}}\to 0 \tag{30}$$

Non conservative Matrix $\widetilde{\mathbf{H}}$

As for the non conservative term $\widetilde{\mathbf{H}}$, we will use the following relation for the component H_{33}, proposed in Rosatti and Fraccarollo (2006) (used also in Rosatti et al, 2008, Rosatti and Begnudelli, 2010) on the basis of physical considerations:

$$H_{33}=-g(1+c_k\Delta_s)\left(h_k-\frac{|z_R-z_L|}{2}\right)(z_R-z_L) \qquad \text{where} \quad k=\begin{cases} L & \text{if } z_L\le z_R \\ R & \text{otherwise} \end{cases} \tag{31}$$

Comparison between SCRS and GCRS

As discussed before, the advantage of the General-Closure Riemann Solver is its generality, because it allows using any closure relationship and because it is very simple to implement in any numerical scheme, making it straightforward to switch from a closure to another one, simply changing the subroutines that compute c and its derivatives $\partial c/\partial h$ and $\partial c/\partial u$. This makes it possible to use different closures depending on the specific needs, for instance depending on the phenomenon, on the type of soil, and so on.

To compare the two approaches, we solve the same problem by adopting the same closure (for example, Eq.10) and using the two methods. In particular, in SCRS we obtain the Roe matrices $\underline{\widetilde{\mathbf{A}}}$ and $\underline{\widetilde{\mathbf{B}}}$ from the matrices $\underline{\mathbf{U}}$ and $\underline{\mathbf{F}}$ where the closure (10) has been plugged in. On the other hand, in GCRS the Roe matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ are derived from the matrices $\underline{\mathbf{U}}$ and $\underline{\mathbf{F}}$ where the concentration is expressed as a generic function of (u,h). Therefore, SCRS the linearization is performed *before* plugging in the closure, while in GCRS it is performed *after*. As a consequence, $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ are in general slightly different from $\underline{\widetilde{\mathbf{A}}}$ and $\underline{\widetilde{\mathbf{B}}}$. However, as a fundamental constraint of the Roe scheme, when $\mathbf{U}_R \to \mathbf{U}_L$, the two matrices must tend to converge to the same value, and the differences between the two approaches tend to vanish.

In this section we compare the two approaches described above (SCRS and GCRS) in terms of the components of **A** and **B**, of the system eigenvalues, and of the results of simulations performed using a numerical model similar to the one described by Rosatti (2008) where SCRS and GCRS are implemented.

Jacobians A and B

Given the left and right states $\mathbf{U}_L$ and $\mathbf{U}_R$, we consider the averages $\tilde{h}=(h_L+h_R)/2$ and $\tilde{u}=(u_L+u_R)/2$. We introduce now the quantities ε_h and ε_u defined as:

$$\begin{cases} h_L=\tilde{h}(1-\varepsilon_h) \\ h_R=\tilde{h}(1+\varepsilon_h) \end{cases} \quad \begin{cases} u_L=\tilde{u}(1-\varepsilon_u) \\ u_R=\tilde{u}(1+\varepsilon_u) \end{cases} \tag{32}$$

We can now express the differences:

$$\tilde{\mathbf{R}}_{\mathbf{B}}=\tilde{\mathbf{B}}-\underline{\tilde{\mathbf{B}}} \quad ; \quad \tilde{\mathbf{R}}_{\mathbf{A}}=\tilde{\mathbf{A}}-\underline{\tilde{\mathbf{A}}} \tag{33}$$

in terms of the averaged variables defined above and of the deviations ε_h , ε_u. We obtain:

$$\tilde{\mathbf{R}}_{\mathbf{B}}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R_{31}^b & R_{32}^b & 0 \end{bmatrix} \quad ; \quad \tilde{\mathbf{R}}_{\mathbf{A}}=\begin{bmatrix} 0 & 0 & 0 \\ R_{21}^a & R_{22}^a & 0 \\ R_{31}^a & R_{32}^a & 0 \end{bmatrix} \tag{34}$$

Where:

$$R_{31}^b=\Delta_s\frac{c_\sim q_\sim}{\tilde{h}}\left[\frac{\varepsilon_u^3}{\varepsilon_h-2\varepsilon_u}\right] \quad ; \quad R_{32}^b=\Delta_s\frac{c_\sim q_\sim}{\tilde{u}}\left[\frac{-\varepsilon_h\varepsilon_u^2}{\varepsilon_h-2\varepsilon_u}\right] \tag{35}$$

and:

$$\begin{gathered} R_{21}^a=\frac{c_\sim q_\sim}{\tilde{h}}\left[\frac{\varepsilon_u^3}{\varepsilon_h-2\varepsilon_u}\right] \quad ; \quad R_{22}^a=\frac{c_\sim q_\sim}{\tilde{u}}\left[\frac{-\varepsilon_h\varepsilon_u^2}{\varepsilon_h-2\varepsilon_u}\right] \\ R_{31}^a=2\Delta_s\frac{c_\sim F_\sim}{\tilde{h}}\left[\frac{\varepsilon_u^3}{\varepsilon_h-2\varepsilon_u}\right] \quad ; \quad R_{32}^a=2\Delta_s\frac{c_\sim F_\sim}{\tilde{u}}\left[\frac{-\varepsilon_h\varepsilon_u^2}{\varepsilon_h-2\varepsilon_u}\right] \end{gathered} \tag{36}$$

Note that the non zero components become singular when:

$$\text{(SC 2)} \quad \varepsilon_h-2\varepsilon_u\to 0 \tag{37}$$

Eigenvalues

Recalling Eqs.(6)-(7), and in consistency with the notation used so far, the eigenvalues corresponding to two approaches are:

$$\begin{aligned} &\text{SCRS}: \det(\underline{\tilde{\mathbf{J}}}-\underline{\tilde{\lambda}}\mathbf{I})=0\to\underline{\tilde{\lambda}}^k \ \text{ where: } \underline{\tilde{\mathbf{J}}}=(\underline{\tilde{\mathbf{A}}}+\underline{\tilde{\tilde{\mathbf{H}}}})\tilde{\mathbf{B}}^{-1} \\ &\text{GCRS}: \det(\tilde{\mathbf{J}}-\tilde{\lambda}\mathbf{I})=0\to\tilde{\lambda}^k \ \text{ where: } \tilde{\mathbf{J}}=(\tilde{\mathbf{A}}+\tilde{\tilde{\mathbf{H}}})\tilde{\mathbf{B}}^{-1} \end{aligned} \tag{7}$$

where k=1,2,3, with the eigenvalues in ascending order. We consider a set of left and right states $\mathbf{U}_L$, $\mathbf{U}_R$ such as $\tilde{h}=1$, $\tilde{u}=1$ and the deviations ε_h and ε_u range in the interval [−0.25;+0.25]. As for the height of the bed step at the interface, we assume $z_R-z_L=0.1$m (but the results show very little variations using different values of the bed step height). The differences between the values of $\tilde{\lambda}^1$ computed with SCRS and GCRS normalized by the exact value are shown in Figure 1.

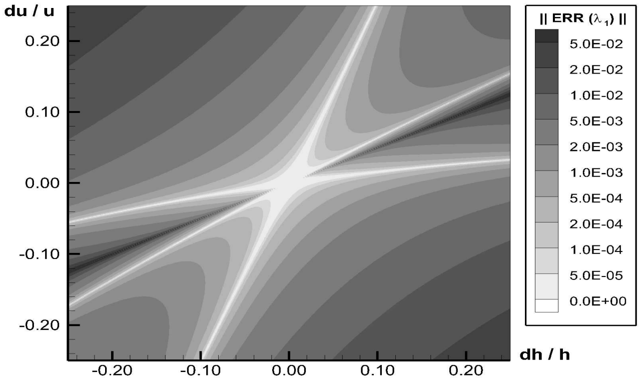

Figure 1: Differences between the values of $\tilde{\lambda}^1$ with SCRS and GCRS, normalized by the exact value of $\tilde{\lambda}^1$.

Singular Terms

We have seen that the terms $\hat{h}$, $\hat{q}$ and $\hat{\Phi}$ become singular under the condition (30), while $\tilde{\mathbf{R}}_{\mathbf{A}}$ and $\tilde{\mathbf{R}}_{\mathbf{B}}$ become singular under the condition (37). Using Eqs. (22), (26) and (32) we obtain that the two conditions are equivalent:

$$\Delta^h\frac{\partial\tilde{c}}{\partial h}+\Delta^u\frac{\partial\tilde{c}}{\partial u}\to 0 \quad \xleftrightarrow{\text{(Eqs. 22,26,32)}} \quad \varepsilon_h-2\varepsilon_u\to 0 \tag{38}$$

Moreover considering the variable $c(u,h)$ and the differential $dc = du(\partial c/\partial u) + dh(\partial c/\partial h)$, a discretization gives the expression that appears in Eq.(30), so the SC becomes: $\Delta^c\to 0$. Using the closure (10) we obtain, neglecting the terms of order $O(\varepsilon^3)$:

$$\text{(SC 3)} \quad \Delta^c=c_L-c_R=2\varepsilon_u-\varepsilon_h\to 0 \tag{39}$$

As a conclusion, we have that the Roe matrices become singular when $c_L=c_R$. In order to avoid instabilities when singular conditions are approached, we use the following fix: for $|c_L-c_R|<10^{-6}$, we substitute the averages that become singular with arithmetic averages. Since left and right conditions are almost coincident, the error is nearly zero.

Application to a dam break problem

Lastly, to compare the two approaches, we consider a dry-bed dam break problem over movable bed. We run the model for different cell-sizes and using the two approaches (SCRS and GCRS), always adopting the closure (10). The channel is 100m long, and the initial position of the dam is x=50m . Initial conditions are: h_0=5m , u_0=0m /s upstream of the dam and dry bed downstream. With regard to the other parameters, we have $g=9.806\text{ms}^{-2}$, $\Delta_s=1.65$ and

β=0.9806m²s^{-1}. The computational grid is composed by 1600 cells. Numerical results are shown in Figure 2. along with the analytical solution. As it can be seen, the numerical solutions corresponding to the two solvers are nearly undistinguishable. The non dimensional error of h and c are reported in Figure 3., where h is scaled by h_0 and c by the maximum concentration c_b.

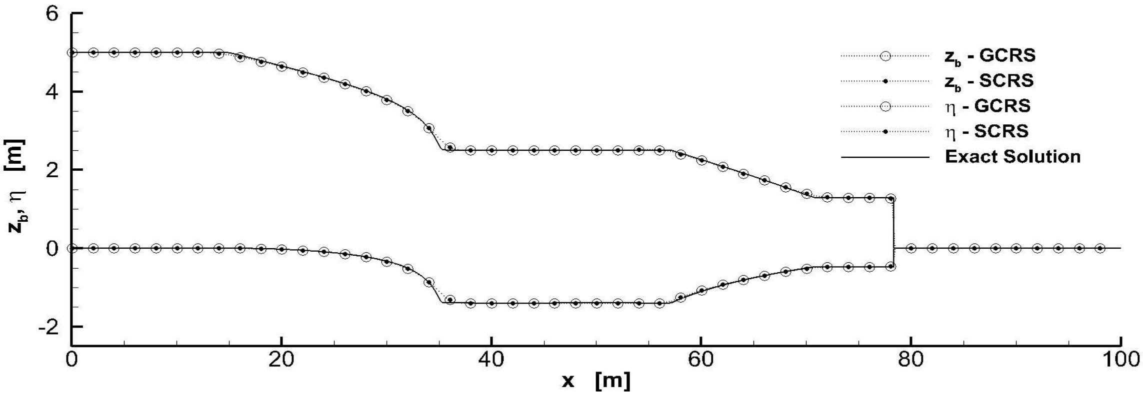

Figure 2: Dry-bed dam break test: numerical results using SCRS and GCRS, along with exact solution.

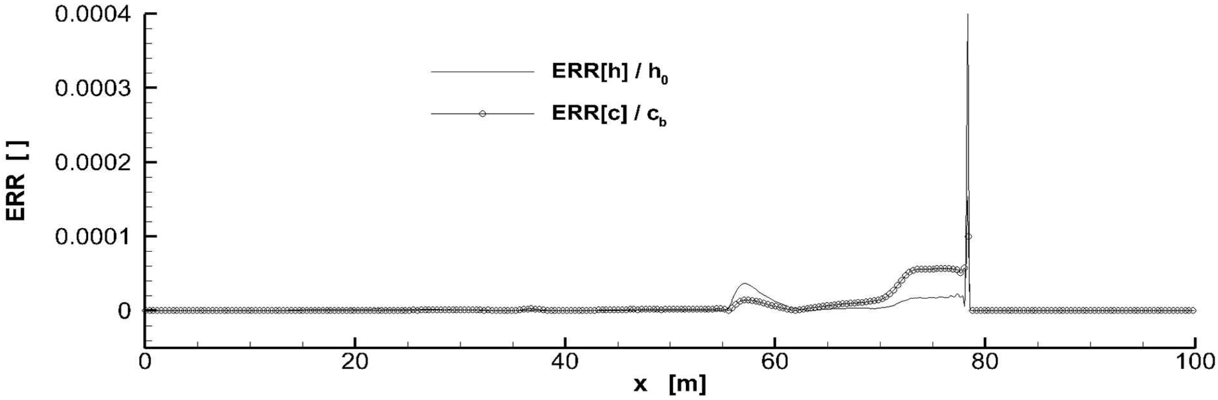

Figure 3: Dry-bed dam break test: non-dimensional deviations of the values of *h* and *c* between SCRS and GCRS.

Conclusions

A new general formulation of the Generalized Roe scheme for hyper-concentrated 1D shallow flows over a mobile bed has been shown. The proposed approach allows to use any possible closure relationships without the need of finding proper Roe matrices (being such a task prohibitive when the closure is not particularly simple). It is shown to be completely general, easy to implement, and as accurate as the standard Roe approach. Due to these characteristics, the method is suitable for application in a wide range of models for sediment transport and debris flow.

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