DYNAMICS OF SUBMERGED GRAVITATIONAL GRANULAR FLOWS

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Abstract

According to the definitions proposed by Takahashi, debris flows are extraordinary mass transport phenomena driven by gravity. To investigate the basic physics of debris flows, it is very useful to analyze the flow of a mixture of identical, spherical particles saturated by water down a steep channel in steady flow condition.
Across the depth we can observed: an external layer, near to the free surface, dominated by nearly instantaneous contacts among the particles (*collisional regime*), an internal region dominated by prolonged contacts among the particles (*frictional regime*) and a *static bed* in which the particles are immobile. Armanini *et al.* (2009) analyzed different rheological mechanisms inside the flow, focusing on the coexistence of frictional and collisional regimes, on the stress transmission inside the flow and on particles kinematics. In particular, it was observed that granular flows may show locally a typical intermittence of the flow regime, switching alternatively from frictional to collisional. In general, the tensor of the granular phase can be assumed to be the composition of two tensors: T_{ij}^{g-coll} represents the stresses exchanged with a collisional mechanism and T_{ij}^{g-fric} represents the stresses expressed by a frictional mechanism. While the rheology of the collisional regimes is well described by the dense gas analogy (*kinetic theory*), a persuasive theoretical description of the frictional regime does not yet exist. A Coulombian scheme is often assumed, but this hypothesis is rather limitative because it requires a constant concentration or a distribution of particles concentration known a priori. An interesting scheme of this kind was recently proposed by GDR-MiDi (2004), but this model does not contain a suitable formulation for the granular pressure (equation of state of the mixture). Following Armanini (2010), we propose a reinterpretation of the model, as weighted average of a pure Coulombian stress (dependent on the static friction angle at the static bed level) and of a dynamic stress, represented by a dynamic friction angle. Besides, a state relation is introduced for the granular pressure and the dynamic friction angle is derived from the kinetic theory. The proposed relations are finally compared with the experimental data.

Rheology of the granular flows

A proper and realistic approach to the problem of the debris flows consists of treated debris flows as a two-phase fluids. One is the interstitial fluid, that follows the fluid mechanic laws with an appropriate rheology, and the other one is the solid phase, which consists of a granular fluid provided also with a specific rheology. If we do not consider the presence of cohesive (clayish) particles in the flow, the solid phase is composed by particles with size bigger than fine sand and can be treated as a granular fluid. Under the hypothesis that the particles dimensions are much smaller than the control volume and that this volume is infinitesimal, the particles could be liken to a fluid with an own rheological law, which describes the interaction mechanism among the grains.
The mechanic of these gravitational granular flows is represented by the equations of the conservation of the mass and the momentum, that are [25],[18]:

$$
\begin{cases}
\dfrac{\partial \rho_m}{\partial t} + \dfrac{\partial\left(\rho_m u_i^m\right)}{\partial x_i} = 0 \\[2ex]
\dfrac{\partial \rho_m u_i^m}{\partial t} + \dfrac{\partial\left(\rho_m u_i^m u_j^m\right)}{\partial x_j} = \rho_m g_i^m + \dfrac{\partial T_{ij}^m}{\partial x_j} + F_i^m
\end{cases}
$$

With $m = f, g$ where: f represents the fluid phase and g the granular phase. In these equations ρ_m is the density of the phases, so: for the fluid phase $\rho_f = (1-c)\rho_w$, where c is the particle concentration and ρ_w is the density of the interstitial liquid, and for the granular phase $\rho_g = c\rho_s$, where ρ_s is the material density of the particle; u_i^m are the components of the velocity of both phases; F_i^m are the components of the force per unit volume that represents the interaction between solid and liquid phase; T_{ij}^m are the components of the stress tensor of phases; g_i^m are the components of the force of mass per unit volume that acts on each phase. Because the flow is governed by gravity, this force coincides with the gravity acceleration and so

$g_i^f = g_i^s = -g\, \partial z / \partial x_i$, where z represents the vertical upward direction.

2D uniform flow of liquid-granular mixture

In the following, we will consider a uniform flow in the longitudinal direction x_i . By summing term by term the momentum equations of the two phases, the interaction forces, F_i^m, will be eliminated. Moreover, if the particles concentration is high enough, it is possible to neglect the term relative to the stresses internal to the fluid phase, τ_i^f , and in the end to write:

$$\frac{\partial \tau_{12}^g}{\partial x_2} = (1 + c\Delta)\rho_w g \frac{\partial z}{\partial x_1} \quad (1)$$

$$\frac{\partial p^g}{\partial x_2} = c\Delta\rho_w g \frac{\partial z}{\partial x_2} \quad (2)$$

where $\Delta = (\rho_s - \rho_w)/\rho_w$ is the submerged relative density of the particles.

Rheology of the granular phase

The rheology of the granular phase can be outlined by two modalities of interaction between particles: almost instantaneous contacts and long lasting contacts. Two regimes correspond to this two interactions, which are termed respectively *collisional regime*, represented by the stresses tensor T_{ij}^{g-coll}, and *frictional regime*, represented by T_{ij}^{g-fric} [16], [7], [13], [10]. It is generally assumed [16], [17], [21] that the two regimes are stratified and so physically separated: the collisional regime in the upper layer of the flow and the frictional regime in the lower layer, where the particle concentration is bigger. On the contrary, recent experimental investigations [3] show that the two regimes are alternated in space and time through a intermittent mechanism, similar to the one that exists between the viscous sub-layer and the turbulent sub-layer in a wall boundary layer. Generally under both hypotheses, it is possible to assume that:

$$T_{ij}^g = T_{ij}^{g-coll} + T_{ij}^{g-fric} \quad (3)$$

The collisional part of the granular tensor is well described by the kinetic theory of granular flows [16], [14], [19], [15], [20], derived by analogy with the kinetic theory of gases, according to which the flow of the particles is similar to the flow of the molecules of an ideal gas. The temperature is replaced by the granular temperature: $\Theta = \langle u_i'^p u_i'^p \rangle / 3$, in which the symbol $\langle\rangle$ represents the average done on all the particles that are in a control volume, small enough compared to the dimensions of the boundary and large enough compared to the particle size. According to this theory, the rheological law represents the collisional regime is expressed as [8]:

$$T_{ij}^{g-coll} = -f_1 \rho_s \Theta \delta_{ij} + \mu^{g-coll}\left(\frac{\partial u_i^g}{\partial x_j} + \frac{\partial u_j^g}{\partial x_i}\right) \quad (4)$$

where T^{g-coll} is the collisional component of the stresses tensor of the granular phase ; δ_{ij} Kronecker's delta. The granular temperature represents the kinetic energy of the collisional flow of the particles, and its behavior is represented by particle kinetic energy balance:

$$\rho_s\left(\frac{\partial \Theta}{\partial t} + u_j^g \frac{\partial \Theta}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\kappa_\Theta \frac{\partial \Theta}{\partial x_j}\right) + \mu_{g-coll}\left(\frac{\partial u_j^g}{\partial x_i} + \frac{\partial u_i^g}{\partial x_j}\right)^2 - f_5 \rho_s \frac{\Theta^{1.5}}{d_p} \quad (5)$$

The expressions of the coefficients are reported in the following table 1:

Table 1: Expression of the coefficients of the kinetic theory.

$$f_1 = (1 + 4c\eta_p g_o)c$$

$$f_2 = \frac{5\sqrt{\pi}}{96\eta_p(2-\eta_p)}\left(1 + \frac{8}{5}\eta_p c g_0\right)\left(\frac{1}{g_0} + \frac{8}{5}\eta_p(3\eta_p - 2)c\right) + \frac{8/5}{\sqrt{\pi}}\eta_p c^2$$

$$f_4 = 25\sqrt{\pi}16\eta_p(41-33\eta_p)(1+125\eta_p c g_0)\left(\frac{1}{g_0} + 125\eta_p^2(4\eta_p - 3)c\right) + 4\sqrt{\pi}\eta_s c^2 g$$

$$f_5 = 12\sqrt{\pi}c^2 g_0(1-e^2)$$

$\mu^{g-coll} = f_2 \rho_s \sqrt{\Theta} d_p$; $k_\Theta = f_4 \rho_s \sqrt{\Theta} d_p$

$g_0(c) = (1 - c/c^*)^{-1/3}$; $e = 0.9 - 2.85 St^{-0.5}$

$St = \rho_s d_p \Theta^{0.5} / 18\mu_w$; $\eta = (1 + e_p)/2$

The system of equations (4) - (5) describes the behavior of the collisional scheme through a fairly convincing theoretical approach, which gave good results in many experimental situations.

On the contrary, the problem of the rheology of the frictional regime is still open, and validated general formulations do not exist yet. In this condition the collisions among particles are not instantaneous, but they become long lasting and they could involve more particles at the same time. In granular flows of heavy materials governed by gravity, under the material that is moving, if the boundary conditions allow it, it is possible to find an immobile layer, because the frictional forces among grains are so high that do not permit any flow. In uniform flow, this condition is identified as an *equilibrium condition* between the granular flow and the immobile bed [4], because there is no net exchange of material between the flow and the bed. Also experiments have shown that the system becomes increasingly frictional [3] while approaching the immobile bed, and on this frontier a Coulombian condition is established. Recently the GDR-MiDi group [9], [10], [11] proposed a rheological model that combines the Coulombian, rate independent scheme with a rate dependent model. This approach was originally formulated for granular dry 2D flows. It is based on the observation, derived by molecular dynamics simulations, that the ratio between the shear stress and the pressure is a function of a single dimensionless parameter I, termed *inertial number* by Da Cruz *et al.* [9], defined as:

$$I = d_p \dot{\gamma} / \sqrt{p^g / \rho_s} \qquad (16)$$

where $\dot{\gamma}$ is the strain rate of the granular flow. The inertial number represents the rate between two temporal scales [9]: a micro scale $d_p / \sqrt{p^g/\rho_s}$ that is the time in which a particle falls in an empty space with dimension of the particle d_p by the action of a pressure p^g; and a macro scale proportional to the local strain rate $\dot{\gamma}$. The GDR-MiDi rheological model can be written as:

$$\frac{\tau_{ij}^g}{p^g} = \tan\varphi^{fric} + (\tan\varphi^{coll} + \tan\varphi^{fric}) \frac{I}{I + I_0} \qquad (17)$$

I_0 is an experimental constant less then 1. One of the main limit of the MiDi formulation is that the state equation was originally not provided. Later a linear relationship between particle concentration c and inertial parameter was suggested [11].

In this paper we have tried to overcome these limits. The first suggestion [2] consists in the introduction of the kinetic theory formulation in the rheological relationship (17). It is possible to observe in fact that eq (17), considering the granular shear stress τ_{ij}^g as a linear combination of a Coulombian stress p^g tan φ^{fric}, with constant friction angle typical of the frictional regime, and a analogous Coulombian stress, in which the friction angle is depending on the shear rate trough the inertial number that in this case is used as weighting factor:

$$(I + I_0)\tau_{ij}^g = p^g \tan\varphi^{fric} I_0 + p^g \tan\varphi^{coll} I \qquad (18)$$

The above equation divided by $(I_o + I)$ corresponds to equation (17). The last term $p^g I/(I_0 + I)\tan\varphi^{coll}$ can be replaced by the corrisponding expression derived by the kinetic theory:

$$\tau_{12}^g = p^g \frac{I_0}{I_0 + I} \tan\varphi^{fric} + \mu^{g-coll} \frac{\partial u_i^g}{\partial x_j} \qquad (19)$$

In addition to the rheologial relationship (19) it is necessary to specify also the equation of state, that is the relationship between pressur and granular temperature. A suitable relationship of this type was proposed in [2], in analogy with equation (19). This formulation was sligtly modified in [1] and numerically checked. Becouse the results of the system did not reproduce in a satisfory way the experimental data, in this paper we have develop the following new formulation :

$$p^g = p^g \frac{I_0}{I_0 + I} + \rho_s f_1 \Theta \qquad (20)$$

The mean difference between this relation and the relation proposed in [1] is that here the frictional component in the equation of state depends directly on the local value of the granular pressure.

It should be noted that in the equations (19) and (20), when the MiDi inertial number I tends to infinite (pure collisional regime), the granular pressure and the shear stress tend to the kinetic one. On the contrary if the inertial number tends to 0, the shear stress will tend to become purely Coulombian and the kinetic component of the pressure vanish.

The system formed by eqs. (1), (2), (5), (19), (20) is well posed and can be numerically integrated provided that the proper boundary conditions are assigned.

Numerical Method

The governing equations of the proposed model can be written under the general form of a nonlinear system of differential algebraic equations (DAE) as follows:

$$\frac{d}{dt}\mathbf{E}(\mathbf{Q}(t)) = \mathbf{f}(\mathbf{Q}(t), t), \quad \mathbf{Q}(0) = \mathbf{Q}_0, \qquad (21)$$

where $\mathbf{Q} = \mathbf{Q}(t) = (q_1(t), q_2(t), ... q_n(t)) \in \mathrm{R}^n$ is the unknown state vector, and $\mathbf{E}(\mathbf{Q}) \in \mathrm{R}^n$ and $\mathbf{f}(\mathbf{Q}, t) \in \mathrm{R}^n$ are two nonlinear functions of the state vector $\mathbf{Q}$ and the independent variable t. $\mathbf{Q}_0$ is the known initial condition of the initial value problem (21). For its numerical solution we use a Galerkin method, based on the following expression for the unknown solution vector:

$$\mathbf{Q_h}(t) = \sum_{l=0}^{N} \theta_l(t)\hat{\mathbf{Q}}_l := \theta_l \hat{\mathbf{Q}}_l, \qquad (22)$$

where $\theta_l(t)$ represent piecewise polynomial basis functions of maximum degree N and $\hat{\mathbf{Q}}_l$ are the unknown coefficients of the numerical solution. In the above relation we have used classical tensor notation with the Einstein summation convention over two equal indices. Equation (22) is valid for one timestep $\Delta t = t^{n+1} - t^n$, where t^n is the current solution time. To obtain the unknown coefficients $\hat{\mathbf{Q}}_l$, the DAE is multiplied with test functions $\theta_k(t)$ that are identical with the basis functions (classical Galerkin approach), and is subsequently integrated over a time step to obtain the following weak formulation of the DAE:

$$\int_{t^n}^{t^{n+1}} \theta_k(t)\left(\frac{d}{dt}\mathbf{E}(\mathbf{Q_h}(t)) - \mathbf{f}(\mathbf{Q_h}(t),t)\right)dt. \qquad (23)$$

For the test and basis functions $\theta_k(t)$ we choose the Lagrange interpolation polynomials that pass through the $N+1$ equidistant Newton-Cotes quadrature points, $t_l^n = t^n + (l-1)/(N-1)\Delta t$, hence we use a *nodal basis*. Therefore, the numerical approximations of the nonlinear functions $\mathbf{E}$ and $\mathbf{f}$ are simply given by

$$\mathbf{E_h}(t) = \theta_l \hat{\mathbf{E}}_l, \quad and \quad \mathbf{f_h}(t) = \theta_l \hat{\mathbf{f}}_l, \qquad (24)$$

with

$$\hat{\mathbf{E}}_l = \mathbf{E}(\hat{\mathbf{Q}}_\mathbf{l}), \quad and \quad \hat{\mathbf{f}}_l = \mathbf{f}(\hat{\mathbf{Q}}_\mathbf{l},, \mathbf{t_l^n}) \qquad (25)$$

due to the choice of the nodal basis. The weak formulation (23) for the unknowns $\hat{\mathbf{Q}}_l$ therefore becomes

$$\left(\int_{t^n}^{t^{n+1}} \theta_k(t)\frac{d}{dt}\theta_l(t)dt\right)\hat{\mathbf{E}}_l = \left(\int_{t^n}^{t^{n+1}} \theta_k(t)\theta_l(t)dt\right)\hat{\mathbf{f}}_l, \qquad (26)$$

or, in a more compact matrix-vector notation:

$$\mathbf{K}_{kl}\mathbf{E}(\hat{\mathbf{Q}}_l) - \mathbf{M}_{kl}\mathbf{f}(\hat{\mathbf{Q}}_l, t_l^n) = 0, \qquad (27)$$

with $\hat{\mathbf{Q}}_0 = \mathbf{Q}(t^n)$, and the mass matrix $\mathbf{M}_{kl}$ and the stiffness matrix $\mathbf{K}_{kl}$, which can both be precomputed once and for all. The resulting nonlinear algebraic equation system (27) of dimension $n(N+1)$ is solved by a standard Newton method for systems with a line-search-type globalization strategy. The initial guess is provided using a second order Crank-Nicholson-type scheme for the DAE (21) to initialize the nodal values $\hat{\mathbf{Q}}_l$ at all time levels t_l^n:

$$\frac{\mathbf{E}(\hat{\mathbf{Q}}_{l+1}) - \mathbf{E}(\hat{\mathbf{Q}}_l)}{t_{l+1}^n - t_l^n} = \frac{1}{2}\left(\mathbf{f}(\hat{\mathbf{Q}}_{l+1}, t_{l+1}^n) + \mathbf{f}(\hat{\mathbf{Q}}_l, t_l^n)\right). \qquad (28)$$

Equation (28) is again a nonlinear algebraic equation system, however, of smaller dimension n, which is again solved by a globally convergent Newton method. The proposed Galerkin-type method (27) is theoretically of arbitrary order of accuracy in the independent variable t and can be used inside a classical *shooting method* for solving DAE boundary value problems of the type

$$\frac{d}{dt}\mathbf{E}(\mathbf{Q}(t)) = \mathbf{f}(\mathbf{Q}(t),t), \quad \mathrm{Q}(t_0) = \mathrm{Q}_0, \mathrm{Q}(t_1) = \mathrm{Q}_1, \qquad (29)$$

where Q_0 and Q_1 are the known boundary values of the boundary value problem (BVP) (29).

Numerical Results

In the next figures, we have reported a comparison between the prediction of the model and some experimental results obtained by Armanini *et al.* [2]. The figures represent the profiles in direction normal to bed of the most important physical variables.

It should be stressed, however, that the only parameter of the model that needs to be calibrated is I_o. We have assumed $I_o = 0.345$. This model catchs better than the model of the system [1] the experimental data, but in the present case the parameter I_o much greater than the one assumed in [1].

All the variable are made properly dimensionless, and in particular the distance from the static bed is expressed by $\eta = x_2/h$, where h is the flow depth. The system is solved according the following boundary conditions assigned at the boundary of the static bed $\eta = 0$:

Table 2: The boundary conditions assigned at the static bed.

u^g	1E-07
c	$0.999c^*$
Θ	1E-06
$du^g/d\eta$	0.07
$d\Theta/d\eta$	1E-08

Figures 1, 2 and 3 show the distributions of the dimensionless granular velocity, granular concentration and temperature respectively.

The model captures the tendency of the experimental data, and the results are reasonably comparable to them just in the proximity of the static bed, that is where the frictional contact are dominant. Near the free surface the results of the model tend to deviate systematically respect to experimental data.

Similar arguments can be made regarding the distributions of the shear stress and of the pressure (Figures 4 and 5). For these parameters the model seems to catch better the experiments, but this is just an apparent agreement, because the relative error on the contrary is definitely bigger.

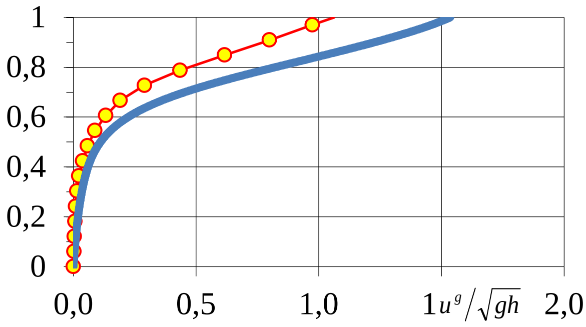

Figure 1: Velocity profile, comparison between results of the numerical simulation and experimental data.

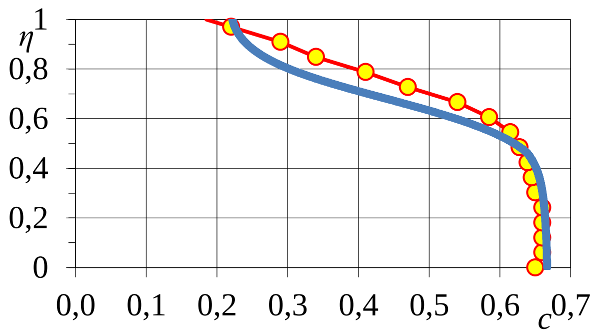

Figure2: Particle concentration profile, comparison between results of the numerical simulation and experimental data

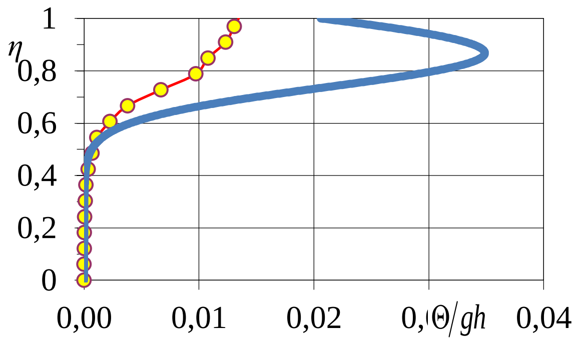

Figure3: Granular temperature profile, comparison between results of the numerical simulation and experimental data.

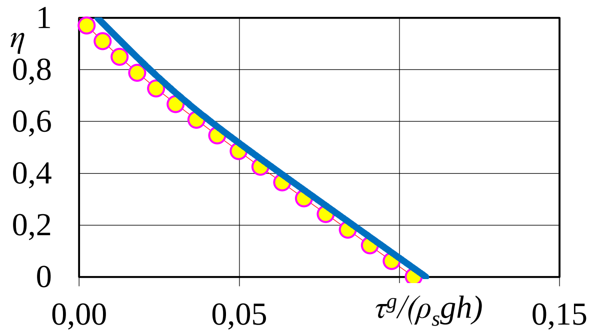

Figure 4: Granular shear stress profile ,comparison between results of the numerical simulation and experimental data.

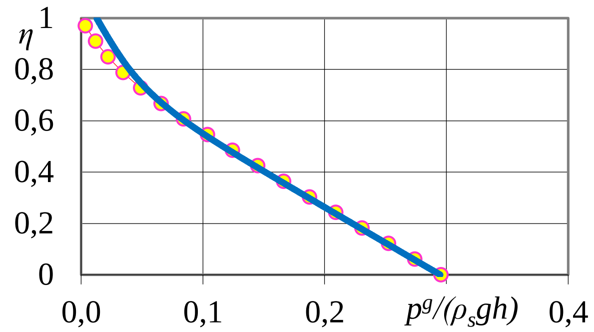

Figure 5: Granular pressure profile, comparison between results of the numerical simulation and experimental data.

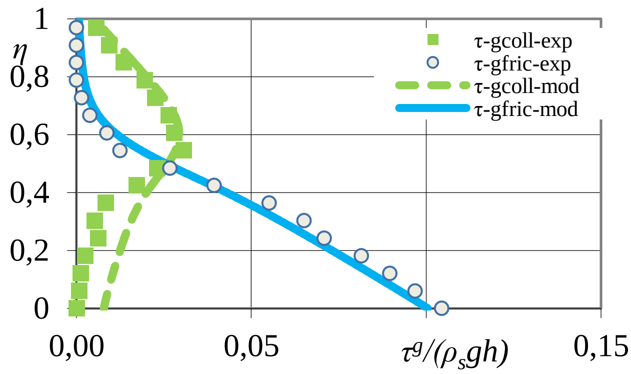

Figure 6: Distributions of the collisional and frictional components of the granular shear stress. Comparison between results of the numerical simulation and experimental data.

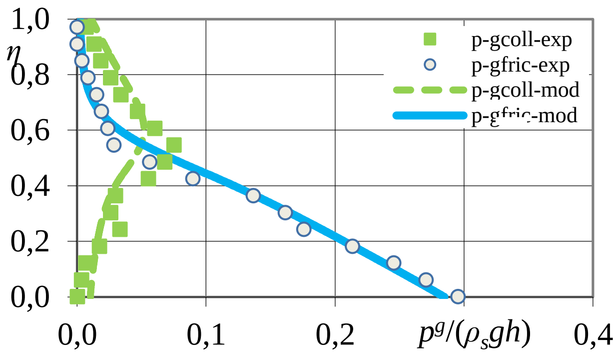

Figure 7: Distributions of the collisional and frictional components of the granular pressure. Comparison between results of the numerical simulation and experimental data.

It must be stressed, however, than we cannot conclude that the frictional model is correct while the collisional one is not. The collisional model in fact is strongly non linear and it is very sensitive to boundary conditions, which are instead determined by the frictional regime. Some preliminary results suggest that we have to reconsider the frictional model, as it is possible to argue also from the Figures 6 and 7 in which we have reported the distribution of the frictional and collisional components of the shear stress and of the pressure. The maximum deviations are in the area next to the static bed, where the frictional regime is dominant, specially for the shear stresses. A similar behavior, but less pronounced, is evident in the distribution of the components of the granular pressure.

Conclusion

In the paper we have presented a rheological model relative to a granular, submerged flow driven by gravity. Respect to the previous schemes, the model considers the simultaneous presence in the flow of the frictional and of the collisional regimes. The model can be considered as an evolution of the GDR-MiDi [11] model and the one proposed in [1]. The results of the model catch in a reasonable way the trend of the experimental data, even if the quantitative comparison cannot be considered satisfactory.

A reconsideration of the GDR-MiDi for the collisional regime will be the next step to do.

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