# THE HYDRAULIC JUMP IN CONVERGING CHANNELS: AN INTEGRAL MECHANICAL BALANCE

**Alessandro Valiani[1] & Valerio Caleffi[1]**

[1]Department of Engineering, University of Ferrara, Italy, via G. Saragat 1, 44122 Ferrara
E-mail: alessandro.valiani@unife.it, valerio.caleffi@unife.it

## Abstract

This paper studies the integral conservation of linear and angular momentum in the steady hydraulic jump in a linearly converging channel, following the recent research line of the authors concerning the same phenomenon in a linearly diverging channel.

The flow is considered divided into a mainstream, that conveys the total liquid discharge, and a roller, where no average mass transport occurs. No macroscopic rheological relationship is assumed, so mass, momentum and angular momentum integral balances are independent relationships. Normal stresses are assumed hydrostatically distributed on each vertical and viscous stresses are assumed negligible with respect to turbulent stresses. Horizontal velocity is considered uniform in the mainstream and horizontal momentum and angular momentum in the roller are neglected with respect to their mainstream counterparts.

Using such simplified assumptions an analytical solution is obtained for the free surface profile of the flow, which is fundamental for finding the sequent depths and their positions. Such solution permits to compute the jump length, which is assumed equal to the roller length. Mainstream and roller thicknesses can also be derived. The model may also be used to theoretically derive the average shear stresses exerted by the roller on the mainstream and the power losses per unit weight. This final relationship, which returns the well-known classical expression for total power loss in the jump, demonstrates the internal consistency of the mechanical model.

## Introduction

The work by Valiani & Caleffi (2011) analyzes the classical, integrated, hydraulic theory of the jump in the linearly diverging channel. The present work is the exact counterpart, when the channel is linearly diverging. As shown in two important experimental works of the sixties (Rubatta, 1963; Rubatta, 1964), the varying width channel increases the number of degrees of freedom of the theoretical problem with respect to the classical prismatic case from one to two. Assuming as prescribed the non-dimensional discharge and the specific energy (downstream/upstream) ratio as independent parameters, we propose here a theoretical method to find the sequent depths, their positions, the free surface profile and mainstream profile, minimizing to one the semi-empirical relationship which must be used in the theory. Such a relation is just an expression giving the proper expansion rate of the mainstream at the beginning of the jump.

In the following, the main hypothesis concerning the properties of the flow field are explained; integral mechanical balances are set up and solved to find the unknowns of the problem. Validation of theoretical results is performed comparing them with experimental results by Rubatta (1964). Finally, we give some conclusions and perspectives on this findings.

## Basic Assumptions on the Physics

We address to Valiani (1997) and Castro-Orgaz & Hager (2009), for a review of the fundamental hypothesis, which are classically assumed in a direct jump. The idealized scheme is that of a shock, conserving mass and linear longitudinal momentum, whilst depth, average velocity and specific energy have abrupt discontinuities. At a refined scale, Valiani (1997) posed the problem of the finite length of the jump, which allows to consider a gradually varying depth and velocity inside; of the non-negligibility of vertical velocity, which stimulate the analysis of vertical momentum balance; of the integral unbalancing of angular momentum (moment of momentum), which can be explained considering the role of vertical momentum fluxes, including also Reynolds stresses. The same problems were faced and analyzed by Valiani & Caleffi (2010, 2011), considering the hydraulic jump in linearly diverging channels.

The analysis is repeated for the linearly converging channel, following step by step the framework of the previous works (Valiani & Caleffi, 2010; 2011). The bottom friction is neglected, to obtain a direct comparison with classical, total force preserving, results on prismatic channels. Viscosity and surface tension effects are both neglected, having in mind a length scale of the problem which is appropriate for hydraulic engineering applications rather than small-scale analysis.

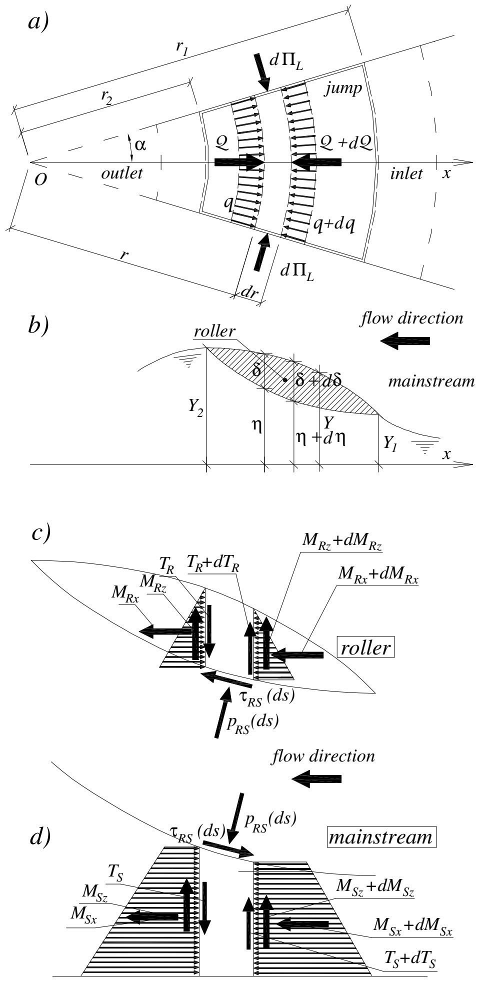


Figure 1: sketch of the diverging channel, geometry and nomenclature. a) plan view, Q and q are generic integral and distributed quantities; b) vertical section; c) control volume of the roller; d) control volume of the mainstream.

The scheme is 2D, axially symmetric, and all results are referred to the unit-angular amplitude channel. The upstream and downstream sections of the jump are considered to remain (on the average) in the same position and to be recognized, at the usual order of approximation typical of the hydraulics textbooks; stability analysis is not considered here.

In a coordinate system ($r$, $\theta$, $z$), the flow is radial, inward directed; $x$ is the axis of symmetry of the flow, and is also the direction of the longitudinal integral forces (Fig. 1). The vertical direction is identified by $z$ axis; $z$ = 0 at the (flat) bottom of the channel, with positive direction against gravity; every physical quantity is assumed independent on $\theta$; radial velocity $v_r$ is considered as positive if directed inward.

The discharge $Q$ is supposed to be constant, assuming an average steady state motion. The angular amplitude of the converging channel is $2\alpha$, as it is sketched in Fig. 1; the discharge for unit arc length is $Q/2\alpha$. The current depth of the channel is $Y$; its upstream value (beginning of the jump) is $Y_1$; its corresponding downstream value, that is the sequent depth, is $Y_2$.

Even neglecting shear stresses on the bottom, the sequent depths are not related each other by an established relationship, as in the prismatic case. In fact, the lateral walls exerts pressure forces on the flow, and the resultant is depending on the free surface profile. For converging channels, the total force decreases in the flow direction (the contrary for diverging channels). We intentionally avoid empirical estimates of such decreasing, just because the free surface profile is a result of the presented mechanical scheme, being evaluated using a proper closure of integral momentum and angular momentum balances.

As discussed extensively by Valiani & Caleffi (2010, 2011), a certain precaution must be used in writing complete (energy, momentum, angular momentum) mechanical balances in a shallow flow, in order to avoid ill-posed schemes (Valiani & Caleffi, 2003). In the following, we write mechanical balances at the simplest order of approximation, being conscious, for example, that vertically hydrostatic pressure distribution is an highly simplified assumption. Notwithstanding this, our conclusions demonstrate – as it is extensively shown in Valiani (1997) and Valiani & Caleffi (2011) – the internal consistency of the adopted mechanical model. The main purpose of such model is not capturing the details, but the ability to preserve integral properties without an excessive sensitiveness to uncertain tuning parameters or particular experimental conditions.

Only the direct jump ($\mathrm{Fr}_1^2 \geq 3$, where $\mathrm{Fr}_1$ is the upstream Froude number) is analyzed. The mean (turbulence effects are averaged over a proper time scale) velocity vector has ($v_r$, $v_z$) components, while $v_\theta = 0$. Outside the hydraulic jump vertical velocities are neglected, and inviscid, specific energy preserving flow is assumed.

Some basic properties of the flow field outside the jump are summarized here. The radial velocity is $v_r = Q/(2\alpha Y)$; the specific energy is: $E = Y + Q^2/[2g(2\alpha r Y)^2]$, where $g$ is gravity; pressure vertical distribution is hydrostatic: $p = \gamma(Y-z)$, where $\gamma$ is the specific weight of the fluid.

The total force of the stream on a whole cross section, directed along the $x$ axis, is: $F_t = \Pi_s + F_d = [2\gamma \sin\alpha]\{r\,Y^2/2 + Q^2/[g(4\alpha^2 r^2 Y)]\}$, where the static term and the dynamic one are put into evidence. If two cross sections are considered, whose distance is d$r$, the elementary force (on the control volume between them) exerted by lateral walls is directed along $x$ axis, and may be evaluated from: $\mathrm{d}\Pi_L = [2\gamma \sin\alpha](Y^2/2)\,\mathrm{d}r$. This force explains the decreasing of the total force in the flow direction if the flow is converging. In fact, if the flow is directed inward, d$r$ is negative and consequently $\mathrm{d}\Pi_L$ is also negative.

Outside the jump, the non-dimensional significant parameter for non-viscous flow is: $\Gamma = Q/\left[2\alpha E^2 (gE)^{1/2}\right]$. Critical depth and critical radius (minimum of specific energy and total force) are defined by: $Y_c = (2/3)E$ and $r_c = Q/\left[2\alpha Y_c (g\,Y_c)\right]^{1/2}$. The longitudinal to vertical length scale ratio is: $r_c / Y_c = (3/2)^{5/2}\Gamma \simeq 2.76\Gamma$. Using such depth and such radius as vertical and horizontal reference scales, the non-dimensional free surface elevation outside the jump is: $\xi = \left[y(3-2y)^{1/2}\right]^{-1}$, being $\xi = r/r_c$ and $y = Y/Y_c$ the non-dimensional current radius and the non-dimensional current depth.

We address Valiani (1997) and Caleffi & Valiani (2010, 2011) for an extensive support of the simplified hypothesis adopted here. The vertical hydrostaticity of normal pressure is also adopted by Engelund (1981), Fredsøe and Deigaard (1992), Adami (1983), Castro Orgaz & Hager (2009). The most important consequence of this assumption is the local balance between gravity forces and bottom reaction forces, so that their resultant over an infinitesimal control volume between two sections at a distance d$r$, is zero; the net angular momentum is zero, too.

Even if viscosity remains responsible of the energy dissipation at small scales (Valiani, 1997), viscous stresses are neglected with respect to turbulent stresses. The macroscopic rheological behavior of the liquid, concerning such stresses, is considered as unknown, so that angular momentum balance is an independent tool in the present analysis.

Such a balance can be satisfied only if vertical turbulent stresses and vertical momentum rising effects are properly taken into account. Interestingly, we can observe that these two effects can be considered as a whole unique term: this is physically correct, being the average turbulent stress related to the average of the product of velocity components fluctuations in $(r, z)$ directions.

Inside the mainstream, a formally uniform velocity distribution occurs (Engelund, 1981; Valiani, 1997; Castro-Orgaz & Hager, 2009; Valiani & Caleffi, 2011). Moreover, any distinction between jump length and roller length is avoided, to circumvent uncertainties between the two definitions, also looking at experimental difficulties to identify the downstream end of the jump (Rubatta, 1963; Rubatta, 1964; Engelund, 1981; Fredsøe and Deigaard, 1992; Valiani, 1997; Castro-Orgaz and Hager, 2009; Valiani & Caleffi, 2011). The length of the jump is here uniquely defined as the distance between the sequent depths that satisfy the longitudinal momentum conservation.

## Mechanical Balances

The flow field is assumed to be divided in two distinct regions: the mainstream and the roller, according a scheme which is used both in coastal engineering and in fluid mechanics (Engelund,1981; Fredsøe and Deigaard, 1992; Valiani, 1997; Castro-Orgaz and Hager, 2009; Valiani & Caleffi, 2010, 2011). The roller is the part of the stream (of thickness $\delta$) which does not contribute to the net transport of mass, due to the inversion of velocity profile inside it. The net mass transport in the mainstream is, on the opposite, equal to the total flow discharge, that is the mainstream thickness $\eta$ satisfies the condition: $Q=(2\alpha r\eta)\, v_r$. The whole depth of the flow in the jump region is $Y = \eta+\delta$, while outside the jump region, where no inversion of the mean velocity profile occurs, the whole depth is $Y = \eta$.

Valiani (1997) for the prismatic channel and Valiani & Caleffi (2010, 2011) for the diverging channel extended the work by Engelund (1981) assuming that the thickness of the mainstream in the jump varies from $Y_1$ to $Y_2$, with a certain growth rate, that is the consequence of balance laws and not imposed a priori; next, the longitudinal momentum crossing the mainstream can be estimated assuming a uniform velocity distribution of the mean horizontal velocity, as $M_{Sx} = [2\gamma \sin\alpha]\ \{Q^2/[g(4\alpha^2 r\eta)]\}$; finally, the momentum crossing the roller, $M_{Rx}$, is negligible with respect to the momentum crossing the mainstream, that is $M_{Rx} << M_{Sx}$.

The line $\eta = \eta(r)$ in the mean vertical plane is a streamline; its elementary length is: $\mathrm{d}s = (\mathrm{d}r^2 + \mathrm{d}\eta^2)^{1/2}$. The mutual interactions between the mainstream and the roller are described by two components of the total stress (Valiani, 1997; Valiani & Caleffi, 2010, 2011): $p_{RS}$ is the normal pressure, $\tau_{RS}$ is the total tangential stress, considered as positive if directed outward, so that the corresponding shear force exerted by the roller on the mainstream acts (as expected) from left to right, being the flow directed from right to the left.

The hydrostatic distribution hypothesis on the vertical implies: $p_{RS} = \gamma\delta$ at the interface. We consider a control volume inside two cylindrical vertical surfaces, located at a distance respectively $r$ and $r$+d$r$ from the origin; its basis is a portion of circular crown, whose angular amplitude is $2\alpha$. The limiting upper surfaces are the free surface elevation and the mainstream-roller boundary. The arc of the corresponding streamline has a length d$s$; mechanical balances are written separately for the mainstream and the roller. All moments over the infinitesimal element in the following are evaluated using the point O, the origin of the reference system, as the pole. Counterclockwise torques are considered as positive.

The pressure forces we are going to consider are: the static force (on the current cross section) related to mainstream flow, $\Pi_S$; to the roller, $\Pi_R$; the static forces exerted by lateral walls on the mainstream and on the roller, $\mathrm{d}\Pi_{LS}$ and $\mathrm{d}\Pi_{LR}$, respectively. They are:

$$\Pi_S = 2\gamma \sin\alpha \left( r \frac{Y^2 - \delta^2}{2} \right); \Pi_R = 2\gamma \sin\alpha \left( \frac{r\delta^2}{2} \right);$$
$$\mathrm{d}\Pi_{LS} = 2\gamma \sin\alpha \left( \frac{Y^2 - \delta^2}{2} \right) \mathrm{d}r; \mathrm{d}\Pi_{LR} = 2\gamma \sin\alpha \left( \frac{\delta^2}{2} \right) \mathrm{d}r \quad (1)$$

The first part of the mainstream pressure forces has an arm of $(Y/3)$; the second part has an arm of $(\eta + \delta/3)$. The mainstream and the roller are supposed to exchange an horizontal force proportional to $p_{RS}$ d$\eta$ = $\gamma\delta$ d$\eta$ and a vertical force proportional to $p_{RS}$ d$r$ = $\gamma\delta$ d$r$ . The forces due to tangential stresses are proportional to $\tau_{RS}$ d$r$ (horizontal) and to $\tau_{RS}$ d$\eta$ (vertical). Vertical mass forces on the mainstream elementary control volume are proportional to $\gamma\eta$ d$r$, on the roller volume to $\gamma\delta$ d$r$, and on the whole depth control volume as $\gamma(\eta+\delta)$ d$r$. Vertical reaction force on the control volume d$R$ is considered as due only to hydrostatic pressures, so that exactly counterbalances the weight being proportional to $\gamma(\eta+\delta)$ d$r$. On the vertical cylindrical cross sections related to the mainstream, to the roller and to the whole depth, vertical shear forces are $T_S$, $T_R$, $T$, respectively. Longitudinal momentum crossing the mainstream, the roller and the whole depth is denoted as $M_S$, $M_R$, $M$; corresponding vertical momenta are $M_{Sz}$, $M_{Rz}$, $M_z$ . It is:

$$dM_S = d\left\{ \left[ 2\gamma \sin\alpha \right] \left[ \frac{Q^2}{g\left(4\alpha^2 r\eta\right)} \right] \right\}; \quad M_R \ll M_S \quad (2)$$

The momentum crossing the mainstream has an arm of $(\eta/2)$, while the second relationship implies that angular momentum crossing the roller is negligible with respect to the angular momentum crossing the mainstream.
By considering the axial symmetry and integrating over the proper arc length, longitudinal integrated quantities are proportional to a factor [2sin($\alpha$) $r$], vertical integrated quantities are proportional to [2$\alpha$ $r$], and the arm of vertical forces is [2$r$ sin($\alpha$)/(2$\alpha$)]. Angular momenta of all the listed forces are described in Valiani & Caleffi (2011) and not repeated here.

**Linear and angular momentum balances**

Momentum balance - $x$ component (mainstream – roller, respectively):

$$\tau_{RS} = +\gamma\eta \frac{\mathrm{d}\eta}{\mathrm{d}r} + \gamma\eta \frac{\mathrm{d}\delta}{\mathrm{d}r} + \rho \frac{Q^2}{4\alpha^2 r} \frac{\mathrm{d}}{\mathrm{d}r}\left( \frac{1}{r\eta} \right)$$
$$\tau_{RS} = -\gamma\delta \frac{\mathrm{d}\eta}{\mathrm{d}r} - \gamma\delta \frac{\mathrm{d}\delta}{\mathrm{d}r} \quad (3)$$

Momentum balance - $x$ component on the whole stream (in the jump):

$$0 = \frac{\mathrm{d}}{\mathrm{d}r}\left[ \frac{1}{2}\gamma(\eta+\delta)^2 \right] + \rho \frac{Q^2}{4\alpha^2 r} \frac{\mathrm{d}}{\mathrm{d}r}\left[ \frac{1}{r\eta} \right] \quad \Leftrightarrow$$
$$\Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}r}\left[ \frac{1}{2}\gamma r(\eta+\delta)^2 + \rho \frac{Q^2}{4\alpha^2}\left( \frac{1}{r\eta} \right) \right] = \frac{1}{2}\gamma(\eta+\delta)^2 \quad (4)$$

Momentum balance - $z$ component (mainstream – roller, respectively):

$$\tau_{RS}\, r \frac{\mathrm{d}\eta}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r} \frac{\left(M_{Sz} - T_S\right)}{2\alpha}$$
$$\tau_{RS}\, r \frac{\mathrm{d}\eta}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}r} \frac{\left(M_{Rz} - T_R\right)}{2\alpha} \quad (5a,b)$$

Momentum balance - $z$ component on the whole stream (in the jump):

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(M_z - T_z\right) = 0 \Leftrightarrow \left(M_z - T_z\right) = \text{const} \quad \Rightarrow$$
$$\Rightarrow \quad \left(M_z - T_z\right) = \left(M_z - T_z\right)_1 = \left(M_z - T_z\right)_2 \quad (6)$$

Angular momentum balance (mainstream - roller respectively):

$$\frac{\left(M_{Sz} - T_S\right)}{2\alpha} = -\tau_{RS}\, r\eta +$$
$$+\frac{1}{2}\gamma r\eta^2 \frac{\mathrm{d}}{\mathrm{d}r}(\eta+\delta) + \rho \frac{Q^2}{4\alpha^2} \frac{\mathrm{d}}{\mathrm{d}r}\left( \frac{1}{2r} \right)$$
$$\frac{\left(M_{Rz} - T_R\right)}{2\alpha} = +\tau_{RS}\, r\eta +$$
$$+\gamma r\eta\delta \frac{\mathrm{d}}{\mathrm{d}r}(\eta+\delta) + \frac{1}{2}\gamma r\delta^2 \frac{\mathrm{d}}{\mathrm{d}r}(\eta+\delta) \quad (7)$$

Angular momentum balance (whole stream):

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left[ \frac{1}{6}\gamma(\eta+\delta)^3 \right] + \frac{\mathrm{d}}{\mathrm{d}r}\left[ \rho \frac{Q^2}{\left(4\alpha^2\right)}\left( \frac{1}{2r} \right) \right] = \frac{\left(M_z - T_z\right)}{2\alpha} \Rightarrow$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}r}\left[ \frac{1}{6}\gamma(\eta+\delta)^3 + \rho \frac{Q^2}{\left(4\alpha^2\right)}\left( \frac{1}{4r^2} \right) \right] = \frac{\text{const}}{r} \quad (8)$$

**Final closure system**

As extensively explained in Valiani & Caleffi (2011), the left hand sides in Eqs. 5(a,b) are candidate to be of the order of neglected terms. For this reason, these relations are not used directly, but just to find Eqs. (6) and (7) where the interaction term between the roller and the mainstream, in any case, disappears. The final solution is written in non-dimensional form to obtain scale-independent results. The critical depth, $Y_c$, is chosen as fundamental vertical length scale and the critical radius, $r_c$, as fundamental horizontal length scale. The relation: $\rho Q^2/(4\alpha^2) = \gamma r_c^2 Y_c^3$ is taken into account; non-dimensional coordinates are $\zeta = \eta / Y_c$; $\sigma = \delta / Y_c$. The independent non-dimensional parameters governing the problem are two (Rubatta, 1963, 1964): the non-dimensional discharge, $\Gamma = Q \big/ \left(2\alpha E_1^2 \sqrt{gE_1}\right)$, and the (downstream/ upstream) energy ratio, $\mathcal{E}_{\mathcal{R}} = E_2/E_1$. Prescribed boundary conditions are: $\sigma(\xi_1) = \sigma(\xi_2) = 0$; $\zeta(\xi_1) = y(\xi_1) = y_1$; $\zeta(\xi_2) = y(\xi_2) = y_2$;, together with:

$$\xi_1 = \left( y_1 \sqrt{3 - 2\, y_1} \right)^{-1}; \quad \xi_2 = \left( y_2 \sqrt{3\,\mathcal{E}_{\mathcal{R}} - 2\, y_2} \right)^{-1}; \qquad \text{(9a,b)}$$

The linear and the angular momentum balances for the whole stream, written in non-dimensional, integral form are:

$$\left( \frac{1}{2}\, \xi_2\, y_2^2 + \frac{1}{\xi_2\, y_2} \right) - \left( \frac{1}{2}\, \xi_1\, y_1^2 + \frac{1}{\xi_1\, y_1} \right) = \int_{\xi_1}^{\xi_2} \frac{1}{2} y^2 d\xi; \; \text{(10a,b)}$$

$$\left[ \frac{1}{6} y_2^3 + \left( \frac{1}{4\xi_2^2} \right) - C_1\, ln\left(\xi_2\right) \right] = \left[ \frac{1}{6} y_1^3 + \left( \frac{1}{4\xi_1^2} \right) - C_1\, ln\left(\xi_1\right) \right]$$

being:

$$y = \left[ y_1^3 + \frac{3}{2}\left( \frac{1}{\xi_1^2} - \frac{1}{\xi^2} \right) + 6\, C_1\, ln\left( \frac{\xi}{\xi_1} \right) \right]^{1/3} \qquad (11)$$

Eq. (11) is the non-dimensional free surface profile. The corresponding expressions for the current non-dimensional mainstream thickness, $\zeta$, and non-dimensional roller thickness, $\sigma$, are:

$$\zeta = \frac{1}{\xi\left[F_1 + \mathcal{I} - F_s\right]} \quad \text{and} \quad \sigma = y - \zeta; \quad \text{where:}$$

$$F_1 = \frac{1}{2}\, \xi_1\, \eta_1^2 + \frac{1}{\xi_1\, \eta_1}; \; \mathcal{I} = \int_{\xi_1}^{\xi} \frac{1}{2}\left[ y\left(t\right)\right]^2 dt; \; F_s = \frac{1}{2}\, \xi\, y^2 \qquad (12)$$

As discussed in Valiani (1997) and Valiani & Caleffi (2011), the scheme implies a discontinuity in the surface profile derivative; that is, a change in the scale at which the phenomenon is observed: from the macroscopic, shallow-water like, scale (vertical lengths much more less than horizontal lengths) to the intermediate scale (vertical lengths less than horizontal lengths, but no more than one order of magnitude). The depth and velocity are gradually varying functions inside the jump, and discontinuity is shifted from the function values to the derivative of the profile: cusps are present both at the beginning and at the end of the jump.

To obtain a well-posed problem, the system (9), (10) need the depth profile to be known: the proper expression is (11). An independent expression concerning the quantity $C_1$ is needed. In Valiani & Caleffi (2011) the physical meaning of this quantity is discussed: it is strictly related to the expansion rate of the submerged jet to which the mainstream can be compared. In fact, from longitudinal momentum balance and angular momentum balance, by a simple derivation we can obtain:

$$C_1 = \frac{1}{2\xi_1\, y_1} \left.\frac{d\zeta}{d\xi}\right|_1 = \frac{2.76\;\Gamma}{2\xi_1\, y_1} \left.\frac{d\eta}{dr}\right|_1 = \frac{2.76\;\Gamma}{2\xi_1\, y_1} \tan\beta_1 \qquad (13)$$

$\beta_1$ is the expansion rate of the submerged jet in the jump. The physics of turbulence is the dominant phenomenon which determines such a value (Hoyt and Sellin, 1989). We expect different expressions for prismatic jump (Valiani, 1997), diverging jump (Valiani & Caleffi, 2011) and converging jump (present paper). Using experimental data by Rubatta (1964), and noting the strict physical relationship between the initial steepness of the jump, the average steepness of the jump and the (downstream/ upstream) energy ratio, we propose here the following expression for the constant $C_1$ (that is negative, just because the flow moves from larger to smaller values of $r$):

$$C_1 = -\,0.11\,\Gamma\,\mathcal{E}_{\mathcal{R}}^{\;-0.79} \qquad (14)$$

The final problem we have obtained is the following. Given a prescribed non-dimensional discharge, $\Gamma$, and a prescribed specific energy ratio, $\mathcal{E}_{\mathcal{R}}$, the system to solve consists of the four Eqs. (9a,b) and (10a,b), in the four unknowns, $y_1$, $y_2$, $\xi_1$ and $\xi_2$. Eq. (14) gives the proper value for the constant $C_1$ and Eq. (11) allows to compute the right hand side integral in Eq. (10a), providing the free surface profile for a couple of boundary conditions.

Such a system can be solved using standard software, for example the MATLAB routine “fsolve”, starting from the following guess values of the non-dimensional sequent depths and positions: $y_{20} = 0.98(3/2)\,\mathcal{E}_{\mathcal{R}}$; $y_{10} = 0.5((1-8\mathrm{Fr}_2^2)^{1/2} - 1)\, y_{20}$; $\xi_{10} = (y_{10}(3-2\, y_{10})^{1/2})^{-1}$ and $\xi_{20} = (y_{20}(3\,\mathcal{E}_{\mathcal{R}} - 2\, y_{20})^{1/2})^{-1}$. Downstream depth guess value is obtained considering as small the kinetic energy with respect to the total specific energy of the subcritical flow. Upstream depth guess value is obtained by the classical expression of sequent depths in prismatic channels. Guess position of both depths are simply given by boundary conditions.

This method is validated against a the selected experimental dataset by Rubatta (1964). It is chosen because there is a complete report of raw data, experiments are completely and exhaustively described, and the laboratory channel has a quite large size (this is particularly important, to be on line with theoretical assumptions). Fig. 2 reports the comparison between predictions and measurements, of upstream and downstream sequent depths and positions.

Agreement is very good, particularly considering the extremely low rate of empirical relationships and tuning parameters. The phenomenological/mathematical model presented herein can give further results, omitted herein for space reasons; it is possible to obtain the estimation of turbulent stress interaction between the mainstream and the roller, $\tau_{RS}$, to check the closure of mechanical balances over each part of the jump (mainstream, roller), to compute the power dissipation in the jump, which analytically satisfies the well-known requirement: $P = \gamma\, Q\, \left(E_1 - E_2\right)$.

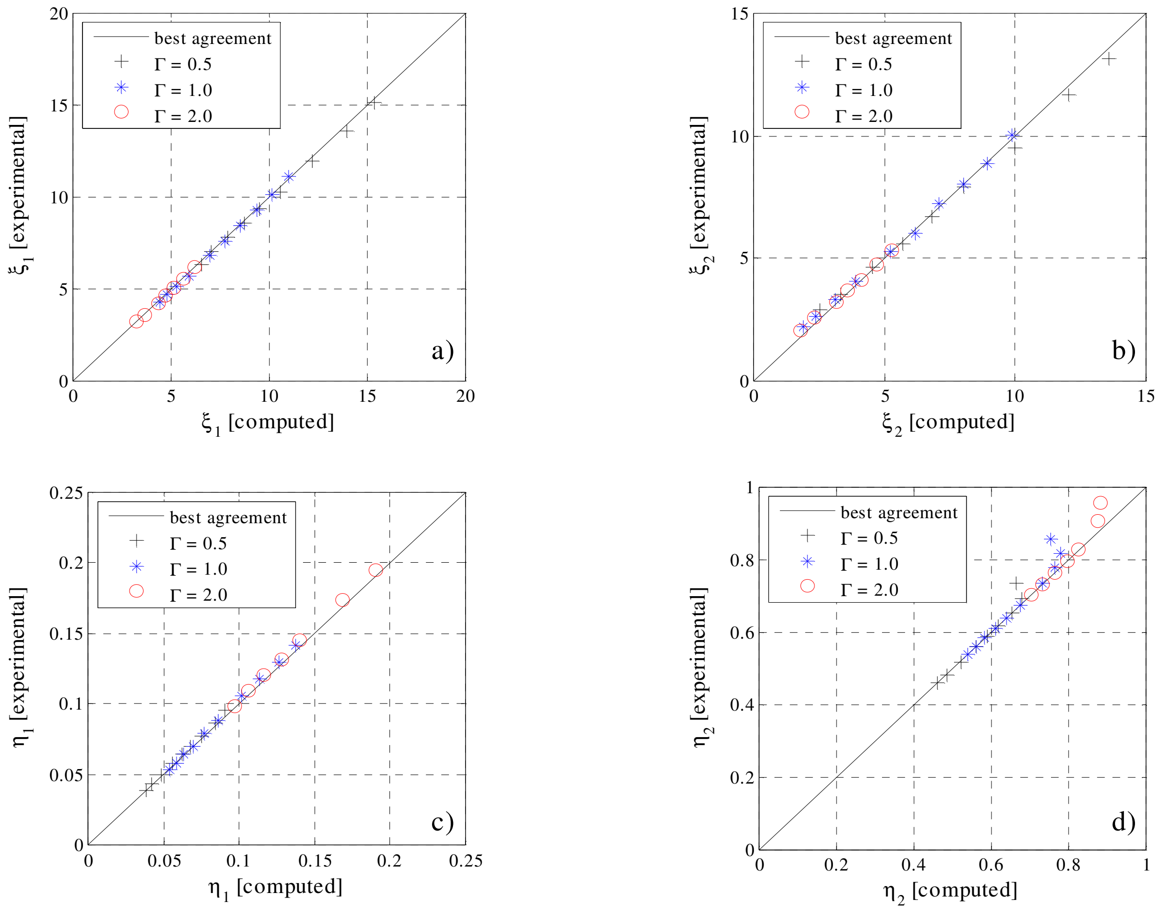


Figure 2: comparison between experimental and computed results: a) upstream jump position; b) downstream jump position; c) upstream sequent depth; d) downstream sequent depth.

## Conclusions

A complete dynamical balance of the hydraulic jump in linearly converging channels is here proposed. Several simplified assumptions are accepted, to catch the substantial aspects of the phenomenon.

Linear (horizontal and vertical) momentum balances and angular momentum balance (around an horizontal axis perpendicular to the channel axis) are written for the mainstream, the roller and the whole flow.

The two independent parameters governing the flow are the non-dimensional discharge and the dissipation rate of the jump. Mechanical balances bring to a final conservation system, consisting of 4 equation in 4 unknowns, the upstream and downstream sequent depths and their positions in the channel. Just one empirical relationship is needed, an expression linking the expansion rate of the mainstream at the beginning of the jump with the (downstream/upstream) energy ratio. Application of this scheme to a selected laboratory channel data base allows a very good validation of the model, which gives results in the expected experimental error range. Free surface profile, roller thickness and mainstream thickness can also be obtained from the present scheme.

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